Efficient System Reliability Analysis of Multi-Layered Soil Slopes Using Multiple Stochastic Response Surfaces

Dian-Qing Li, M.ASCE; Shui-Hua Jiang; Xiao-Hui Qi; and Zi-Jun Cao

Abstract: This paper aims to propose an efficient approach for evaluating the system reliability of multi-layered soil slopes using representative slip surfaces and multiple stochastic response surfaces (SRSs). First, the representative slip surfaces are identified from a large number of potential slip surfaces. For each representative slip surface, a stochastic response surface using the Hermite polynomial chaos expansion is constructed to estimate its factor of safety (FS). Second, direct Monte-Carlo simulations are performed to compute the system failure probability of the slope, of which the minimum FS for each random sample is calculated using SRSs of representative slip surfaces. Finally, a three-layered clay slope is investigated to demonstrate the effectiveness of the proposed approach. The results indicate that the proposed approach can effectively identify the representative slip surfaces of multi-layered soil slopes and produce accurate system failure probability which is commonly at relatively low levels. In addition, the proposed approach does not need to calculate the correlations between different potential slip surfaces for identification of the representative slip surfaces. The system failure probability of a multi-layered soil slope could be significantly underestimated if only the critical slip surface or insufficient representative slip surfaces are used.

INTRODUCTION

Inherent spatial variability of geotechnical properties has been considered as one of the major sources of uncertainty in geotechnical engineering (e.g., Vanmarcke 1977, Phoon et al. 1999). It affects significantly the slope stability (e.g., Cho 2010, Huang et al. 2010, Wang et al. 2011, Li et al. 2014, Jiang et al. 2014, 2015; Jiang and Huang 2016). The slope can be viewed as a series system from a probabilistic point of view (Cornell, 1967, Ditlevsen, 1979, Chowdhury & Xu, 1995), by considering each potential slip surface to be a component and the critical slip surface (CSS) to be the weakest one. The previous studies have demonstrated that there might exist multiple dominating failure modes in a soil slope and these failure modes shall be considered rationally. Most previous studies focused on various slope failure modes caused by stratification (i.e., layered soils) (e.g., Chowdhury & Xu 1995, Zhang et al. 2011, 2013, Ji & Low 2012, Kang et al. 2015) or the inherent spatial variability of soil properties in a single-layered soil (Wang et al. 2011, Li et al. 2013, Jiang et al. 2015). However, few attempts have been made to study system reliability of multi-layered soil slopes considering the spatial variability of soil properties. How to
This paper develops a multiple stochastic response surfaces approach to efficiently evaluate system reliability of multi-layered soil slopes consisting of spatially variable soils. The proposed approach allows explicit modeling of spatial variability of soil properties and performing system reliability analysis by using multiple stochastic response surfaces constructed on representative slip surfaces (RSSs). The paper starts with the identification of representative slip surfaces from a large number of potential slip surfaces and construction of the respective stochastic response surfaces. Then, an implementation procedure of the proposed approach is described. Finally, the proposed approach is illustrated through an example of three-layered clay slope, and parametric studies are performed to explore the effect of inherent spatial variability on the system reliability of the three-layered clay slope in spatially variable soils.

MULTIPLE STOCHASTIC RESPONSE SURFACES APPROACH

To ensure that the slip surface with $FS_{\text{min}}$ (i.e., CSS) is properly located, a large number ($N_s$) of potential slip surfaces are usually considered in deterministic slope stability analysis. The value of $N_s$ is frequently on the order of magnitude of $10^3$ to $10^4$ (e.g., Zhang et al. 2011, Li et al. 2013). It is well recognized that the CSS is searched and identified among the large number of potential slip surfaces for each random sample, which demands considerable computational costs. Such a drawback of MCS becomes even more profound for slope reliability analysis at small probability levels in spatially variable soils because the CSS varies spatially (Wang et al. 2011). To address this problem, several previous studies (e.g., Zhang et al. 2011, 2013, Li et al. 2013) have suggested using some representative slip surfaces, which dominate the slope failure, as a surrogate of the large number of potential slip surfaces to evaluate $FS_{\text{min}}$ for each random sample. The number ($N_r$) of RSSs is generally much less than that (i.e., $N_s$) of potential slip surfaces. Therefore, the CSS can be identified among RSSs with relative ease, and the $FS_{\text{min}}$ can be calculated more efficiently. This subsequently leads to efficient evaluation of system reliability of slope in spatially variable soils.

To select $N_r$ RSSs among the $N_s$ potential slip surfaces effectively, a simple procedure is presented in Jiang et al. (2015). The procedure starts with generating $N_p$ realizations of random fields involved in slope reliability analysis by using Latin hypercube sampling (LHS) and random fields discretization methods such as Karhunen-Loève (KL) expansion (Phoon et al. 2002, Cho, 2010, Jiang et al. 2014, 2015). With respect to determining the $N_p$ value, please see section “Discussion” in Jiang et al. (2015) for details. For each $N_p$ realization of random fields, deterministic slope stability analysis is performed to calculate the $FS$ values for all potential slip surfaces and locate the CSS with $FS_{\text{min}}$. This is repeated for $N_p$ times, leading to $N_p$ CSSs. These $N_p$ CSSs are used as RSSs in this study, where $N_r$ is often less than or equal to $N_p$ because different realizations of random fields might result in the same CSS. In addition, the repeated calculations for the $N_p$ realizations of random fields, simultaneously, lead to $N_p$ $FS$ values for each RSS. It can be observed that, in comparison with the approaches proposed by Zhang et al. (2011) and Li et al. (2013), the procedure can avoid the tedious calculations of correlations different potential slip surfaces for identification of RSSs in system reliability analysis of slope stability.

To further improve the efficiency of slope system reliability analysis at small probability levels, a stochastic response surface is constructed for each RSS to calculate its $FS$ (Li et al. 2011), by which the $FS_{\text{min}}$ for each random sample can be solved instantaneously using explicit functions. This study applies a Hermite polynomial chaos expansion to construct a stochastic response surface for each RSS. Using the Hermite polynomial chaos expansion, the $FS$ for a given RSS is calculated as (e.g., Li et al. 2011, Jiang et al. 2015)
in which \( j_r = 1, 2, \ldots, N_r \), \( N_r \) is the number of RSSs; \( N \) is the total number of random variables in standard normal space corresponding to those used to discretize all random fields; \( a_0, a_{i_1}, a_{i_2}, a_{i_1,i_2}, \ldots \) are the unknown coefficients; \( \Gamma_{j_p}() \), \( j_p = 1, 2, 3, \ldots \) are Hermite polynomials with \( j_p \) degrees of freedom (Li et al. 2011, Jiang et al. 2015); \( \xi = (\xi_1, \xi_2, \ldots, \xi_N) \) is a set of independent standard normal random variables.

For the \( n_{\text{HPCE}} \)-th order Hermite polynomial chaos expansion, there are a total of \((N + n_{\text{HPCE}})!/(N! \times n_{\text{HPCE}}!))\) unknown coefficients (i.e., \( a_0, a_{i_1}, a_{i_2}, a_{i_1,i_2}, \ldots \)) in Eq. (1), which are needed to be determined for construction of the stochastic response surface. Determination of these unknown coefficients just uses \( N_p \) realizations of the random variables (or random fields) and the corresponding \( N_p \) FS values for each RSS as obtained previously. Based on the \( N_p \) random samples and the corresponding FS values for a given RSS, a system of \( N_p \) linear equations is obtained using Eq. (1). Then, a regression-based approach is used to compute the unknown coefficients for the given RSS (Li et al. 2011). After that, the stochastic response surface for the RSS concerned is obtained. It should be mentioned that the accuracy of the stochastic response surface relies on the order (i.e., \( n_{\text{HPCE}} \)) of Hermite polynomial chaos expansion, the number (i.e., \( N \)) of random variables involved in reliability analysis, and the accuracy of estimated coefficients. By this means, the selection of RSSs and construction of the respective stochastic response surface are achieved simultaneously in the proposed approach.

Similarly, the \( N_r \) stochastic responses surfaces are obtained and, collectively, used as a surrogate of the deterministic slope stability analysis with the consideration of uncertainties, to efficiently evaluate \( FS_{\text{min}} \) for each random sample. For example, the \( FS_{\text{min}} \) for the \( k \)-th random sample is calculated as

\[
FS_{\text{min}}^{(k)} = \min_{j_r = 1, 2, \ldots, N_r} FS_{j_r}[\xi^{(k)}]
\]

After that, direct Monte-Carlo simulations (MCS) with a total of \( N_r \) random samples are performed to calculate the system failure probability \( (P_{f,s}) \),

\[
P_{f,s} = \frac{1}{N_r} \sum_{k=1}^{N_r} I\{FS_{\text{min}}^{(k)} < 1.0\}
\]

where \( I\{FS_{\text{min}}^{(k)} < 1.0\} \) is an indicator function. For a given random sample, \( I\{FS_{\text{min}}^{(k)} < 1.0\} \) is taken as the value of 1 when \( FS_{\text{min}}^{(k)} < 1.0 \). Otherwise, it is equal to zero. In this way, the computational costs used for the MCS are minimal and negligible because it is performed using explicit functions between the FS values and input uncertain parameters.

**IMPLEMENTATION PROCEDURE FOR MULTIPLE STOCHASTIC RESPONSE SURFACES APPROACH**

In general, the implementation procedure of the proposed multiple stochastic response surfaces approach involves 4 steps. Details of each step are summarized as follows:
(1) Determine input information for the system reliability analysis, including, but not limited to, slope geometry and statistics (e.g., mean, standard deviation, marginal distribution, cross-correlation coefficient, and autocorrelation function) of soil properties;

(2) Generate $N_p$ realizations of random fields by using the LHS and KL expansion method according to the prescribed statistical information, perform deterministic slope stability analysis with the $N_p$ realizations of random fields to determine $N_r$ RSSs and calculate $N_p$ $FS$ values for each RSS;

(3) Construct $N_r$ stochastic response surfaces for the $N_r$ RSSs based on the $N_p$ samples and the corresponding $N_p$ $FS$ values for each RSS using the Hermite polynomial chaos expansion;

(4) Perform a direct MCS run with $N_t$ random samples to estimate the system failure probability using Eq. (3), in which the $FS_{\text{min}}$ for each random sample is determined using the $N_r$ stochastic response surfaces obtained in step (3).

**ILLUSTRATIVE EXAMPLE: APPLICATION TO A THREE-LAYERED CLAY SLOPE**

A three-layered clay slope example which is adopted from Zhang et al. (2013) and Kang et al. (2015), is investigated in this section to illustrate the proposed multiple stochastic response surfaces approach. As shown in Figure 1, the slope has a height of 6 m and a slope angle of 18.4°. The three soil layers extend to 13.5 m below the top of the slope and have the same total unit weight of 18 kN/m³. The undrained shear strength parameters for these three soil layers are considered as lognormally distributed random fields. In the upper soil layer, the random field $c_{u1}$ has a mean value of 18 kPa (i.e., $\mu_{c_{u1}} = 18$ kPa) and coefficient of variation (COV) of 0.3 (i.e., $COV_{c_{u1}} = 0.3$). In the middle soil layer, the random field $c_{u2}$ has a mean value of 20 kPa (i.e., $\mu_{c_{u2}} = 20$ kPa) and COV of 0.2 (i.e., $COV_{c_{u2}} = 0.2$). In the lower soil layer, the random field $c_{u3}$ has a mean value of 25 kPa (i.e., $\mu_{c_{u3}} = 25$ kPa) and COV of 0.3 (i.e., $COV_{c_{u3}} = 0.3$). The method reported in Li et al. (2015) is adopted here to simulate globally nonstationary random fields of three-layered soil undrained shear strength parameters. The covariance between any two points in different regions is assumed to be zero. In this study, a squared exponential autocorrelation function is used (Jiang et al., 2014, 2015). A horizontal autocorrelation distance $\theta_{h}$ of 20 m, and a vertical autocorrelation distance $\theta_{v}$ of 2.0 m are treated as a reference case. The random fields $c_{u1}$, $c_{u2}$ and $c_{u3}$ are discretized into 348, 580 and 640 elements with a side length of 0.5625 m (see Figure 1) for their realizations, respectively, and the random samples are then generated at the centroid of each element using the KL expansion method. To ensure that the ratio of the expected energy is larger than 95% (Jiang et al., 2014), the number of KL expansion terms to be retained is taken as 10, 10 and 10, respectively. As a reference, the nominal value of $FS_{\text{min}}$ is calculated as 1.285 using Bishop’s simplified method when the mean values of soil properties are used, which is almost identical to the value (i.e., 1.282) reported in Kang et al. (2015). In addition, the critical deterministic slip surface (CDSS) in the nominal case is located and is shown in Figure 1.

Figure 1. The example of a three-layered clay slope.
For identification of RSSs and construction of the respective stochastic response surfaces, 1000 (i.e., $N_p = 1000$) realizations of $c_{u1}$, $c_{u2}$, and $c_{u3}$, are simulated, respectively, and the corresponding CSSs are determined through deterministic slope stability analysis using Bishop’s simplified method. These obtained CSSs are then taken as RSSs in this example, resulting in a total of 78 RSSs, as shown in Figure 2. Note that the CDSS (see the dashed line in Figure 1) is also included in 78 RSSs. After the 78 RSSs are obtained, one stochastic response surface is constructed for each RSS using the second order Hermite polynomial chaos expansion according to the procedure as described previously. This results in 78 stochastic response surfaces. To validate the $FS_{\text{min}}$ calculated from the stochastic response surfaces, the $FS_{\text{min}}$ values obtained from the stochastic response surfaces and the original deterministic analysis (e.g., Bishop’s simplified method) of slope stability using 100 sets of random samples are compared as shown in Figure 3. Note that the $FS_{\text{min}}$ values obtained from the two approaches agree well with each other. This indicates that the stochastic response surfaces are good enough to obtain the $FS_{\text{min}}$ for each random sample in this example.

![Figure 2. The three-layered clay slope with 78 RSSs.](image)

![Figure 3. Validation of stochastic response surfaces.](image)

Based on these stochastic response surfaces, a direct MCS run with 500,000 random samples is performed to calculate $P_{f,s}$ herein. Although a relatively large number (i.e., 500,000) of random samples are generated during MCS, the computational costs used for the MCS are minimal and negligible because the $FS_{\text{min}}$ for each random sample is estimated using explicit functions (i.e., 78 stochastic response surfaces). The $P_{f,s}$ estimated from the proposed approach is 0.13, as shown in
Table 1, which is in good agreement with that (i.e., 0.138) obtained from the LHS using 1000 random samples and Bishop’s simplified method. It seems not necessary to construct the stochastic response surfaces and perform the MCS with stochastic response surfaces to calculate $P_{f,s}$ in this case. However, this is not true for the cases with relatively small values of $P_{f,s}$, e.g., $P_{f,s} < 0.001$.

Table 1. Reliability analysis results ($COV_{c_1} = 0.3$, $COV_{c_2} = 0.2$, $COV_{c_3} = 0.3$).

<table>
<thead>
<tr>
<th>Method</th>
<th>Failure probability</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>RSSs + Stochastic Response Surfaces + MCS</td>
<td>0.13</td>
<td>This study</td>
</tr>
<tr>
<td>Limit Equilibrium Method + LHS (1000)</td>
<td>0.138</td>
<td>This study</td>
</tr>
<tr>
<td>CDSS + Stochastic Response Surface + MCS</td>
<td>0.057</td>
<td>This study</td>
</tr>
</tbody>
</table>

Table 2. Reliability analysis results ($COV_{c_1} = 0.3$, $COV_{c_2} = 0.1$, $COV_{c_3} = 0.1$).

<table>
<thead>
<tr>
<th>Method</th>
<th>Failure probability</th>
<th>Source</th>
</tr>
</thead>
<tbody>
<tr>
<td>RSSs + Stochastic Response Surfaces + MCS</td>
<td>5.28×10^{-4}</td>
<td>This study</td>
</tr>
<tr>
<td>Limit Equilibrium Method + LHS (1000)</td>
<td>0</td>
<td>This study</td>
</tr>
<tr>
<td>Limit Equilibrium Method + LHS (30,000)</td>
<td>7.33×10^{-4}</td>
<td>This study</td>
</tr>
<tr>
<td>CDSS + Stochastic Response Surface + MCS</td>
<td>2.0×10^{-6}</td>
<td>This study</td>
</tr>
</tbody>
</table>

For instance, as both the $COV_{c_2}$ and $COV_{c_3}$ decrease to 0.1 in this example, $P_{f,s}$ decreases significantly and is calculated as $5.28\times10^{-4}$ using the proposed approach (see Table 2). During the calculation, 1000 random samples are, again, generated using the LHS for construction of stochastic response surfaces and a MCS run with 500,000 random samples is performed to obtain $P_{f,s}$ based on the stochastic response surfaces. Note that there is no failure sample among the 1000 random samples generated by the LHS. In other words, 1000 random samples are not sufficient to calculate $P_{f,s}$ in this case because $P_{f,s}$ (i.e., $5.28\times10^{-4}$) is relatively small. To validate the $P_{f,s}$ obtained from the proposed approach, the LHS with 30,000 samples are performed to re-calculate the $P_{f,s}$, in which Bishop’s simplified method is used to calculate $FS_{\min}$ for each random sample. The resulting $P_{f,s}$ is $7.33\times10^{-4}$, which compares favorably with that (e.g., $5.28\times10^{-4}$) estimated from the proposed approach. However, the efforts used for 30,000 realizations of random fields and evaluations of $FS_{\min}$ using Bishop’s simplified method in direct implementation of LHS are much larger than those for the proposed approach. Such good agreement validates the proposed approach and indicates that the reduced series system composed of RSSs represents the three-layered clay slope reasonably well.

In addition, based on the stochastic response surface of the CDSS in the nominal case, the failure probability of CDSS is also estimated using the proposed approach. They are 0.057 and $2.0\times10^{-6}$ (see Tables 1 and 2), respectively, which are apparently less than $P_{f,s}$ (i.e., 0.13 and $5.28\times10^{-4}$). The failure probability of slope stability is significantly underestimated when only the CDSS is considered in slope reliability analysis for the multi-layered clay slope.

With the aid of improved computational efficiency offered by the proposed approach, effect of vertical spatial variability on slope system reliability is explored through a parametric study. The parametric study is performed with vertical autocorrelation distance ($\theta_{ln,v}$) varying from 0.6 to 3.0 m, which are consistent with the typical ranges of $\theta_{ln,v}$ reported in Phoon et al. (1999). For each $\theta_{ln,v}$, the proposed approach is applied to determine the RSSs and the corresponding stochastic response surfaces and then to estimate the system failure probability of slope and failure probability of CDSS. Figure 4 shows the variation of $P_{f,s}$ estimated from the proposed approach as a function of $\theta_{ln,v}$ by a line with squares. The results are obtained at $\theta_{ln,h} = 20$ m and $\theta_{ln,v}$ varying from 0.6 to 3.0 m. The $P_{f,s}$ increases from 3.52% to 15.5% as the $\theta_{ln,v}$ increases from 0.6 to 3.0 m. Overestimation of vertical spatial correlation leads to a significant overestimation of $P_{f,s}$ at small probability levels.
Additionally, the effect of spatial variability on $P_{f,s}$ can be investigated from a system analysis point of view using the proposed approach. The estimated $P_{f,s}$ relies on two factors: the failure probability of each component and the number $N_r$ of components in the reduced series system. Generally speaking, the $P_{f,s}$ increases as the failure probability of each RSS and the number of RSSs increase. Figure 4 shows the variation of the failure probability of CDSS as a function of $\theta_{ln\,v}$ by a line with circles. As the $\theta_{ln\,v}$ increases, the failure probability of CDSS increases. On the other hand, Figure 5 shows the variations of $N_r$ as a function of $\theta_{ln\,v}$. As the $\theta_{ln\,v}$ increases from 0.6 to 3.0 m, $N_r$ is more or less approximately 80. The variation of $N_r$ with the change of vertical spatial correlation is relatively minor. However, as shown in Figure 4, $P_{f,s}$ increases significantly as the $\theta_{ln\,v}$ increases from 0.6 to 3.0 m. Such a significant increase in $P_{f,s}$ is mainly attributed to the increase of failure probability of each RSS as the $\theta_{ln\,v}$ increases.
CONCLUSIONS

The paper developed a multiple stochastic response surfaces approach to efficiently evaluate the system reliability of multi-layered soil slopes in spatially variable soils. The proposed approach facilitates the slope system reliability analysis using multiple stochastic response surfaces constructed on representative slip surfaces (RSSs). The determination of RSSs and construction of the respective stochastic response surfaces are achieved simultaneously. No additional computational efforts are needed for identification of RSSs in the proposed approach. Based on the RSSs and their corresponding stochastic response surfaces, $P_{fs}$ is then evaluated using direct Monte-Carlo simulations with negligible computation costs. In addition, the proposed approach allows gaining insights into the effect of spatial variability on $P_{fs}$ from a system analysis point of view.

ACKNOWLEDGMENTS

This work was supported by the National Natural Science Foundation of China (Project Nos. 51225903, 51329901, 51509125) and the Natural Science Foundation of Hubei Province of China (Project No. 2014CFA001).

REFERENCES


