Research Paper

Bivariate distribution of shear strength parameters using copulas and its impact on geotechnical system reliability

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Abstract

The objective of this paper is to investigate the effect of copulas for constructing the bivariate distribution of shear strength parameters on system reliability of geotechnical structures. First, the bivariate distribution of shear strength parameters is constructed using copulas. Second, the implementation procedure of system reliability analysis using direct Monte Carlo simulation (MCS) is developed. Finally, the system reliability of a retaining wall and a rock wedge slope is presented to explore the effect of copula selection on geotechnical system reliability. The results indicate that the system reliability of geotechnical structures under incomplete probability information could not be determined uniquely because the bivariate distribution of cohesion and friction angle with given marginal distributions and correlation coefficient could not be determined uniquely. The copulas for modeling dependence structure between cohesion and friction angle have a significant influence on the system reliability of geotechnical structures. Such an influence includes two phases separately. The first phase is that the dependence structure between shear strength parameters characterized by copulas affects the reliability of single failure mode, depending on the marginal distributions, dependence structure between shear strength parameters, and reliability level of each failure mode. The second phase is that the reliability of each failure mode influences on system reliability, only depending on reliability level of each failure mode and correlations among various failure modes. It is important to distinguish between the effect of copula selection on reliability of each failure mode and that on geotechnical system reliability.

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1. Introduction

The shear strength parameters (cohesion (c) and friction angle (φ)) are important parameters for reliability analysis of geotechnical structures, such as slopes, retaining walls, and strip footings [1,2,6,12–14,23,26,27,32]. To achieve a realistic evaluation of geotechnical reliability, the joint cumulative distribution function (CDF) or probability density function (PDF) of the shear strength parameters should be known. In geotechnical engineering practice, however, the joint CDF or PDF is often unknown due to limited data from field test or laboratory test. On the basis of these limited data, only the marginal distributions and correlation coefficient underlying the shear strength parameters can be determined, which are referred to as incomplete probability information in this study. It is concluded that the joint probability distribution of shear strength parameters under incomplete probability information could not be determined uniquely [10,46,52].

Recently, copula theory (e.g., [43]) has been applied to construct the joint probability distribution of correlated geotechnical parameters under incomplete probability information. For example, Li et al. [33,34] used the copula approach to construct the joint PDF of two curve-fitting parameters underlying load–displacement curves of piles. Uzielli and Mayne [47,48] investigated the dependence between load–displacement model parameters underlying vertically loaded shallow footings on sands using copula. Tang et al. [46] investigated the impact of copula selection on slope and retaining wall reliability. Wu [49] employed the Gaussian and Frank copulas to model the trivariate distribution among cohesion, friction angle and unit weight of soils. Huffman and Stuedlein [19] adopted several copulas for modeling the measured dependence structure between the coefficients of the two-parameter
Summary of the adopted bivariate copula functions in this study.

Table 1

<table>
<thead>
<tr>
<th>Copula</th>
<th>Copula function, $C(u_1, u_2; \theta)$</th>
<th>Copula density function, $D(u_1, u_2; \theta)$</th>
<th>Generator function, $\phi(t)$</th>
<th>Range of $\theta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>$\Phi_1^{-1}(u_1), \Phi_2^{-1}(u_2))$</td>
<td>$\frac{1}{\sqrt{1-\rho^2}} \exp \left[ -\frac{(u_1 - \mu_1)^2 + (u_2 - \mu_2)^2}{2 \sigma_1^2 \sigma_2^2 (1-\rho^2)} \right]$</td>
<td>$S_1 = \Phi_1^{-1}(u_1)$</td>
<td>$[-1, 1]$</td>
</tr>
<tr>
<td>Plackett</td>
<td>$\frac{S}{\Phi_1^{-1}(u_1) - \Phi_2^{-1}(u_2)}$</td>
<td>$\frac{\Phi_1^{-1}(u_1) - \Phi_2^{-1}(u_2)}{\Phi_1^{-1}(u_1) - \Phi_2^{-1}(u_2)}$</td>
<td>$S_2 = \Phi_2^{-1}(u_2)$</td>
<td>$(0, \infty)$</td>
</tr>
<tr>
<td>Frank</td>
<td>$\frac{1}{2} \ln \frac{1 + \rho u_1 u_2}{1 - \rho u_1 u_2}$</td>
<td>$-\ln \left( \frac{1 - \rho^2}{(1 - \rho^2)^2 - 4 \rho^2 u_1 u_2} \right)$</td>
<td>$S_1 = \Phi_1^{-1}(u_1)$</td>
<td>$(-\infty, \infty)$</td>
</tr>
<tr>
<td>No.16</td>
<td>$\frac{1}{2} (1 + \sqrt{3} \sqrt{u_1 u_2})$</td>
<td>$\frac{1}{2} (1 + \sqrt{3} \sqrt{u_1 u_2})$</td>
<td>$S_2 = \Phi_2^{-1}(u_2)$</td>
<td>$(0, \infty)$</td>
</tr>
</tbody>
</table>

Note: The symbol $\Phi^{-1}$ denotes that the generator function is not available; $\Phi^{-1}$ is the inverse standard normal distribution function; $\Phi_1$ is the bivariate standard normal distribution function with Pearson linear correlation coefficient $\theta$. 

bearing pressure–displacement model. Tang et al. [44] proposed three copula-based approaches to evaluate slope reliability under incomplete probability information. These studies indicate that the copulas provide a fairly general method for constructing multivariate distributions that satisfy some non-parametric measure of dependence and the prescribed marginal distributions.

Constructing the multivariate distributions of correlated geotechnical parameters using copulas is an important step for evaluating reliability of geotechnical structures. The impact of the multivariate distributions of correlated geotechnical parameters using various copulas on geotechnical reliability may be of interest. For this reason, Li et al. [33] explored the influence of copula selection on serviceability limit state reliability of piles. Tang et al. [46] investigated the effect of copula selection on reliability of an infinite slope and a retaining wall. These studies concluded that the copula selection has a significant influence on the probabilities of failure of geotechnical structures. Note that the aforementioned studies only focused on the reliability of single failure mode under the geotechnical structures. However, geotechnical structural systems usually consist of more than one failure mode (e.g., [7,18,20–22,35,40,51]). It is evident that the reliability of single failure mode under geotechnical structures could not represent the system reliability of the geotechnical structures. It is of practical interest to distinguish between the reliability of each failure mode and the system reliability of the entire geotechnical structural system. With these in mind, the effect of copulas for modeling the joint distribution of shear strength parameters on the system reliability of geotechnical structures should be explored, and is the topic of the present research.

This paper aims to explore the effect of copulas for constructing the bivariate distribution of shear strength parameters on system reliability of geotechnical structures under incomplete probability information. First, the bivariate distribution of shear strength parameters is constructed in the copula framework. Four copulas, namely Gaussian, Plackett, Frank, and No.16 copulas (e.g., [43]), are selected to model the dependence structure between cohesion and friction angle. Second, the implementation procedure of system reliability calculation using direct Monte Carlo simulation (MCS) is developed. Finally, the system reliability of a retaining wall and a rock wedge slope is presented to demonstrate the effect of copula selection on geotechnical system reliability.

2. Bivariate distribution of shear strength parameters using copulas

As mentioned in the introduction, this paper will adopt copulas for modeling the bivariate distribution of shear strength parameters. Copulas are functions that couple a multivariate distribution function to its one-dimensional marginal distribution functions. Alternatively, copulas are multivariate distribution functions whose one-dimensional marginal distribution functions are uniform on the interval of $[0, 1]$ (e.g., [43]). There are many copulas in the literature such as Gaussian, t, Plackett, Frank, Gumbel and Clayton copulas. Each copula is characterized by its own dependence structure. According to Sklar’s theorem (e.g., [43]), the bivariate distribution, $F(c, \phi')$, of the two shear strength parameters $c'$ and $\phi'$ can be expressed in terms of a copula function $C(u_1, u_2; \theta)$ and the marginal distributions $u_1 = F_1(c')$ and $u_2 = F_2(\phi')$:

$$
F(c', \phi') = F_1(c')f_2(\phi')D(F_1(c'), F_2(\phi'); \theta)
$$

(1)

where $\theta$ is a copula parameter describing the dependency between $c'$ and $\phi'$. From Eq. (1), the bivariate PDF, $f(c', \phi')$, of $c'$ and $\phi'$ can be obtained as (e.g., [43])

$$
f(c', \phi') = f_1(c)f_2(\phi)D(F_1(c), F_2(\phi); \theta)
$$

(2)

where $D(F_1(c'), F_2(\phi); \theta) = D(u_1, u_2; \theta) = \partial^2 C(u_1, u_2; \theta) / \partial u_1 \partial u_2$.

It is evident that both the copula function $C(u_1, u_2; \theta)$ and the copula density function $D(u_1, u_2; \theta)$ are related to the copula parameter $\theta$. The copula parameter $\theta$ can be determined through the Pearson correlation coefficient $\rho$ between $c'$ and $\phi'$. According to the definition of Pearson correlation coefficient (e.g., [4]), the integral relationship between $\rho$ and $\theta$ can be expressed as follows:

$$
\rho = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{(c - \mu_c)(\phi - \mu_{\phi})}{\sigma_c \sigma_{\phi}} f_1(c)f_2(\phi)D(F_1(c), F_2(\phi); \theta)dc'd\phi'
$$

(4)

where $\mu_c$ and $\mu_{\phi}$ are the means of $c'$ and $\phi'$, respectively; $\sigma_c$ and $\sigma_{\phi}$ are the standard deviations of $c'$ and $\phi'$, respectively. For prescribed marginal distributions of $c'$ and $\phi'$, and correlation coefficient $\rho$ between $c'$ and $\phi'$, the preceding integral equation can be solved iteratively to obtain $\theta$. For instance, a two-dimensional Gaussian–Hermite integral technique, developed by Li et al. [36], can be used for such a purpose.

When the probability information on shear strength parameters is only limited to marginal distributions and a correlation coefficient, a large number of copulas that are consistent with such information can be used to characterize the dependence structure. Since there exists a negative correlation between $c'$ and $\phi'$ (e.g., [31,37,45,46]), the copulas that allow a wide range of negative correlation coefficients are selected to characterize the dependence between $c'$ and $\phi'$. A review of the literature reveals that the Gaussian copula, Plackett copula, Frank copula and No.16 copula (e.g., [43]) are appropriate for describing the dependence structure.
between \( c \) and \( \phi \). The aforementioned four copulas, along with the domains of the \( \theta \) parameter are summarized in Table 1. Among the four copulas, the Gaussian copula is an elliptical copula. The Plackett copula is a member of the Plackett copula family. The Frank and No.16 copulas are commonly used Archimedean copulas. The aforementioned copulas except the No.16 copula are symmetric copulas. No.16 copula is approximately symmetric when the negative correlation is strong. All the four copulas can describe negative dependences, and the values of the correlation coefficients between \( c \) and \( \phi \) can approach –1. Such features are very suitable for describing the dependence structure between \( c \) and \( \phi \).

In order to examine the adequacy of the aforementioned four copulas, measured data of cohesion and friction angle obtained from laboratory tests in Xiaolangdi Hydropower Station in China are used, as shown in Table 2 [53]. Two data sets of effective shear strength parameters are obtained from Consolidated-Drained triaxial compression test (CD) and Consolidated-Undrained triaxial compression test (CU), respectively [50]. More detailed information of the measured data of shear strength parameters can be referred to Yu [50]. A data set of shear strength parameters from CU with 64 data is used for illustration because it has larger sample size.

To visualize the dependence structure underlying the data of shear strength parameters, the measured data (\( c, \phi \)) are plotted in both original space and rank space. Note that the marginal distributions of \( c \) and \( \phi \) are assumed to be normal distributions. The mean and standard deviation of \( c \) are 66 kPa and 29 kPa, respectively. For \( \phi \), they are 22° and 3.4°, respectively. Fig. 1 compares the measured data (64 samples with blue color) and the simulated data (300 samples with red color) of \( c \) and \( \phi \) based on the fitted Gaussian, Plackett, Frank, and No.16 copulas. It can be observed that the measured data \((c, \phi)\) exhibit a significant negative correlation. All the copulas selected could capture the dependency underlying the measured data in original space. To clearly show the dependence underlying the measured data considered, the data in rank space are often used for such a purpose. Then, the measured data in original space are transformed into the standard uniform random vector \( U = (U_1, U_2) \) [41]. The empirical distributions of \( c \) and \( \phi \) are adopted in this study. \( U_1 \) and \( U_2 \) are defined as [41,43]

\[
\begin{align*}
U_1 &= \text{rank}(c) / K, \quad i = 1, 2, \ldots, N \\
U_2 &= \text{rank}(\phi) / K
\end{align*}
\]

in which rank \((c_i)\) (or rank \((\phi_i)\)) denotes the rank of \(c_i\) (or \(\phi_i\)) among the list \([c_1, c_2, \ldots, c_N] \) (or \([\phi_1, \phi_2, \ldots, \phi_N]\)) in an ascending order. The scatter plots for \(U_1\) versus \(U_2\) are shown in Fig. 2. Note that there is a strongly negative dependence between \(U_1\) and \(U_2\). Furthermore, the samples of \(U_1\) and \(U_2\) are basically symmetrical with respect to the diagonal line of a unit square. Applying the algorithms for the simulation of copulas [33], Fig. 2 also compares the measured data and the simulated data of \(U_1\) and \(U_2\) based on the four copulas. It shows that all the copulas selected can capture the symmetry of the measured data about the diagonal line in rank space. From visual inspection, it is very difficult to distinguish the best-fit copula among them due to the limited number of measured data whether in original or rank space.

3. Implementation procedure of system reliability evaluation using MCS

In geotechnical practice, geotechnical structures with multiple failure modes, such as slopes, retaining walls and strip footings are often considered as a series system [6,8,35,38]. For a series system with \( k \) failure modes, occurrence of any failure mode would lead to system failure. To formulate the series system reliability problem, let \( g_i(X) \) \( (i = 1, 2, \ldots, k) \) denote the performance function corresponding to the \( i \)-th failure mode, then the probability of failure of a series system, \( p_{fs} \), is expressed as

\[
pfs = P(g_1(X) \leq 0 \cup g_2(X) \leq 0 \cup \ldots \cup g_k(X) \leq 0)
\]
where \( P(\cdot) \) denotes the probability; \( X = [X_1, X_2, ..., X_n] \) represents the vector of basic random variables involved in the series system reliability problem. Note that the random vector \( X \) may include other random variables such as geometrical parameters or loads in addition to the shear strength parameters \( c' \) and \( \phi ' \) investigated in this study. Note that the performance function \( g_i(X) = F_i(X) - 1 \) is adopted for most geotechnical reliability problems where \( F_i(X) \) is the factor of safety corresponding to the \( i \)-th failure mode. It is widely accepted that MCS [18,21] is an efficient approach for solving the series system reliability problem. In this study, direct MCS method is employed to evaluate the system reliability of geotechnical structures. To facilitate the understanding of the implementation procedure for system reliability calculation, Fig. 3 shows the flowchart of series system reliability analysis using MCS. This procedure consists of three steps. Details of each step are summarized as follows: 

1. Draw \( m \) physical samples \( X_{m.n} = [X_1, X_2, ..., X_n] \) from the joint probability density function \( f(x_1, x_2, ..., x_n) \) of \( X \). This process can be further divided into three steps. First, \( m \) samples of independent standard uniform variables \( V_{m.n} = [V_1, V_2, ..., V_n] \) are generated using MATLAB: \( V_{m.n} = \text{rand}(m, n) \). Second, the vector \( V_{m.n} \) are transformed into the correlated standard uniform variables \( U_{m.n} = [U_1, U_2, ..., U_n] \) for a specified copula. The algorithms for transforming an independent standard uniform vector \( V \) into a correlated standard uniform vector \( U \) for a specified copula are referred to McNeil et al. [42] or Nelsen [43]. Li et al. [33] presented the algorithms for converting \( V \) to \( U \) for the selected four bivariate copulas step by step, which will be used in this study.

Third, the physical samples \( X_{m.n} = [X_1, X_2, ..., X_n] \) can be easily obtained using the isoprobabilistic transformation [4,33]. Set \( U_1 = F_1(X_1), U_2 = F_2(X_2), ..., U_n = F_n(X_n) \), then \( X_{m.n} = [X_1, X_2, ..., X_n] = [F_1^{-1}(U_1), F_2^{-1}(U_2), ..., F_n^{-1}(U_n)] \) in which \( F_i^{-1}(\cdot), F_2^{-1}(\cdot), ..., F_n^{-1}(\cdot) \) are the inverse CDFs of \( X_1, X_2, ..., X_n \), respectively. 

2. Substitute the physical samples \( X_{m.n} \) into the performance functions \( g_i(X), g_2(X), ..., g_n(X) \) corresponding to failure modes 1, 2, ..., \( k \), respectively. For a specified sample \( X_{j.n} \) \( (j = 1, 2, ..., m) \), the system failure will occur if any of \( g_i(X_{j.n}) \) is less than or equal to zero.

3. Count the number (Num) that the series system failure occurs over \( m \) samples. Then, the probability of failure for the series system is obtained as \( p_{fs} = \frac{\text{Num}}{m} \).

System reliability bound methods such as Cornell bounds [8] and Ditlevsen bounds [11] are often used to evaluate system reliability. Compared with the Cornell bounds, the Ditlevsen bounds gives a narrower estimate of the system probability of failure. However, the Ditlevsen bounds need to evaluate the joint failure probabilities of each pair of failure modes, which is tedious for geotechnical practitioners [3]. Hence, the Cornell bounds receive wide applications in geotechnical engineering. For a series system with \( k \) failure modes, the Cornell bounds for system probability of failure is given by [8]

\[
\max(p_{j}) \leq p_{fs} \leq 1 - \prod_{i=1}^{k} (1 - p_{j})
\]  

(7)
where $p_i$ is the failure probability of the $i$-th failure mode. It is clear that the Cornell bounds only require the failure probability of each failure mode, which is relatively simple for geotechnical engineers. Thus, the Cornell bounds method is employed to verify the system probabilities of failure produced by various copulas.

It should be noted that the impact of copulas for modeling bivariate distribution of $(c', \phi')$ on system reliability of geotechnical structures includes two different phases. The first phase is that the correlation structure between $c'$ and $\phi'$ influences the reliability of single failure mode underlying the geotechnical structures. Such an influence is mainly attributed to the difference in bivariate models of $(c', \phi')$ constructed by different copulas. The second phase is that the reliability of each failure mode influences the system reliability of the geotechnical structures. Such an influence is mainly dependent on the correlation among different failure modes and the reliability level of single failure mode. Tang et al. [46] studied the effect of copula selection on single failure mode and found that probabilities of failure produced by different copulas can differ significantly. This study mainly focuses on the effect of copula selection on system reliability in the second phase through two illustrative examples.

4. Illustrative examples

4.1. Example I—System reliability of retaining wall

The first example focuses on the system reliability of a semi-gravity retaining wall, the back of which is vertical and smooth, as shown in Fig. 4. In geotechnical engineering practice, three failure modes underlying the retaining wall should be considered in the design of a semi-gravity retaining wall: (1) sliding along its base (referred to as sliding failure mode), (2) overturning of the wall about its toe (referred to as overturning failure mode), and (3) bearing capacity failure of the foundation soil (referred to as bearing capacity failure mode). The existing deterministic approach evaluates a lumped factor of safety against sliding along the wall’s base as [38]

$$F_{S1} = \frac{c_{int}(a + b) + (W_1 + W_2) \tan \phi_{int}}{P_0}$$

(8)

where $c_{int}$ and $\phi_{int}$ are cohesion and friction angle of interface shear strength, respectively. They are obtained using $c_{int} = 1/3c'_{base}$ and $\phi_{int} = 2/3\phi_{base}$ [5], in which $c'_{base}$ and $\phi_{base}$ are cohesion and friction angle of foundation soil.

Similarly, the factor of safety against overturning failure about the wall’s toe is given by [38]

$$F_{S2} = \frac{M_{resisting}}{M_{overturning}} = \frac{W_1 \times Arm_1 + W_2 \times Arm_2}{P_0 \times Arm_a}$$

(9)

where $W_1$ and $W_2$ are the component weights of the retaining wall, with horizontal lever distances $Arm_1$ and $Arm_2$, respectively, measured from the toe of the wall; $M_{resisting}$ and $M_{overturning}$ denote the actual resisting moments and overturning moments, respectively; $P_0$ is the active earth thrust with a vertical lever distance $Arm_a$. Since the back of the retaining wall is assumed to be vertical and smooth, Rankine analysis is applied to calculating $P_0$. For backfill with cohesion $c'$ and internal friction angle $\phi'$, $P_0$ is given by

$$P_0 = \frac{1}{2} \mu_{soil} H^2 K_a - 2c' H \sqrt{K_a} + \frac{2c'^2}{\mu_{soil}}$$

(10)
in which \( K_a \) is the coefficient of active earth pressure; \( \gamma_{soil} \) is the unit weight of the backfill; \( H \) is the height of the wall. According to Rankine’s theory, \( K_a \) is expressed as

\[ K_a = \tan^2 \left( \frac{\pi}{4} - \frac{\phi}{2} \right) \] (11)

The active earth thrust \( P_a \) will act at a height of \( H_0 \) above the base of the wall with a horizontal direction, which can be expressed as

\[ H_0 = \frac{1}{3} \left( H - \frac{2c}{\gamma_{soil} K_a} \right) \] (12)

In Fig. 4, the following equations can be established, for a wall with a vertical back:

\[ W_1 = \frac{1}{2} \gamma_{wall} b H, \ Arm_1 = \frac{2}{3} b, \ W_2 = \gamma_{wall} a H, \ Arm_2 = b + \frac{a}{2} \] (13)
in which \( \gamma_{wall} \) is the unit weight of the retaining wall concrete.

For the bearing capacity failure mode, the factor of safety is expressed as

\[ FS_3 = \frac{q_a}{q_{max}} \] (14)
in which \( q_a \) is bearing capacity of foundation; \( q_{max} \) is maximum value of soil pressure applied by the wall, which is given by

\[ q_{max,\min} = \frac{W_1 + W_2}{a + b} \left( 1 \pm \frac{6e}{a + b} \right) \] (15)

where \( e \) is the eccentricity of the loads’ resultant with respect to the centerline of the base, which is calculated by

\[ e = \frac{b - \frac{M_{max,\min}}{W_1 + W_2}}{W_1 + W_2} \]

It should be pointed out that the following condition should be satisfied to avoid tensile stress in base of retaining wall:

\[ |e| \leq \frac{a + b}{6} \] (16)

Since the shear strength parameters \( c' \) and \( \phi' \) of the retained soil have a significant influence on the reliability of retaining wall [1,46], both \( c' \) and \( \phi' \) are treated as random variables. It is assumed that \( c' \) and \( \phi' \) follow lognormal distributions [25]. For illustrative purpose, the mean and coefficient of variation (COV) of \( c' \) are assumed to be 14 kPa and 0.4, respectively. The mean and COV of \( \phi' \) are assumed to be 25° and 0.2, respectively. Note that the statistics of \( c' \) and \( \phi' \) are assumed in this hypothetical example because no “real” values of soil statistics are available for a hypothetical case. This shall serve the purpose of illustration well and not significantly affect the general trend of results and conclusions drawn from this study. However, it is also worthwhile to be noted that using the statistics of soil parameters derived from real data in geotechnical reliability analysis is always a prudent choice in design practice, provided that the real data are available and sufficient from the project at hand. In addition, the other eight parameters, namely \( H, a, b, \gamma_{soil}, \gamma_{wall}, \gamma_{base}, \gamma_{c,base} \) and \( \phi_{base} \) are also assumed as constants so that the negative correlation between \( c' \) and \( \phi' \) can be studied without interference from other random variables. The deterministic parameters are \( H = 6.5 \) m, \( a = 0.5 \) m, \( b = 1.45 \) m, \( \gamma_{soil} = 18 \) kN/m\(^3\), \( \gamma_{wall} = 24 \) kN/m\(^3\), \( \gamma_{base} = 18 \) kN/m\(^3\), \( \gamma_{c,base} = 100 \) kPa and \( \phi_{base} = 30^\circ \).
4.1.1. System reliability results of retaining wall

The three performance functions corresponding to the sliding failure and overturning failure of the retaining wall can be expressed as

\[ g_1(X) = F_{S_1}(X) - 1, \quad g_2(X) = F_{S_2}(X) - 1, \quad g_3(X) = F_{S_3}(X) - 1, \]

respectively. \( F_{S_1}(X), F_{S_2}(X), \) and \( F_{S_3}(X) \) are calculated by Eq. (8), Eqs. (9) and (14), respectively. For the example studied, the random vector \( X \) only involves the shear strength parameters \((c', \phi')\) of the retained soil. The least number of samples required for direct MCS, \( N_p \), to evaluate \( p_f \) accurately is calculated by

\[ N_p \geq \frac{1}{p_f/\text{COV}_{p_f}} \]

for a target \( \text{COV}_{p_f} \) (e.g., [4]). For a target \( \text{COV}_{p_f} = 0.1 \), the direct MCS with a sample size of \( 10^8 \) is sufficient to evaluate a probability of failure up to \( 10^{-6} \).

The effect of copulas on the system probability of failure is studied systematically based on two factors: (1) \( \rho \) between \( c' \) and \( \phi' \), and (2) COV scaling factor, \( \lambda \), defined as: \( \lambda = 0.4/\text{COV}_{c'} \) and \( \lambda = 0.2/\text{COV}_{\phi'} \). In parametric studies shown in Fig. 5, each factor varies over a range of values shown in the figure caption while the other parameters remain unchanged.

Fig. 5(a) and (b) shows the system probabilities of failure on log scale for various values of \( \rho \) and \( \lambda \), respectively. Note that all the probabilities of failure are expressed on log scale in this study. It can be observed that the system probabilities of failure decrease as the negative correlation between \( c' \) and \( \phi' \) becomes stronger. As expected, the system probabilities of failure decrease with decreasing \( \text{COV}_{c'} \) and \( \text{COV}_{\phi'} \). The system probabilities of failure produced by different copulas can differ considerably, which indicates that the system reliability of the retaining wall cannot be estimated uniquely. Furthermore, the difference in system probabilities of failure associated with the selected four copulas significantly increases with decreasing system probability of failure. For \( \rho = -0.75 \), the system probability of failure produced by the No.16 copula is 3065 times larger than that produced by the Gaussian copula. Among the selected four copulas, the No.16 copula results in the largest system probability of failure, while the Gaussian copula leads to the smallest system probability of failure.

Therefore, the Gaussian copula, often used for modeling the bivariate distribution of correlated shear strength parameters, will significantly overestimate the system reliability of the retaining wall, which is unconservative for retaining wall safety assessment.

To further investigate the influence of copula selection on the system reliability of the retaining wall, Fig. 6 shows the probabilities of failure for both single failure mode and system of the retaining wall. Four subplots in Fig. 6 correspond to the selected four copulas. Generally, there exists a slight difference between the probabilities of failure for the sliding failure mode and the overturning failure mode. Except for the Gaussian copula, the probability of failure for the bearing capacity failure mode is significantly smaller than those for the sliding failure mode and the overturning failure mode, especially for the No.16 copula as shown in Fig. 5(d). Both the probabilities of failure for the single failure mode and the system decrease with decreasing values of \( \rho \). Although the system probabilities of failure associated with different copulas exhibit large variation, all of them fall within the corresponding lower and upper bounds of Cornell’s method. These results imply that the system probabilities of failure associated with different copulas can be effectively evaluated using MCS.

In order to reflect the difference in system probabilities of failure produced by different copulas in a quantitative way, Table 3 summarizes the relative differences in probabilities of failure produced by different copulas. In Table 3, the probability of failure associated with the Gaussian copula is taken as a reference case. The values in Table 3 are calculated by \( p_f/\text{COV}_{\text{Gaussian}} \) in which \( p_f \) is the probability of failure produced by the Plackett copula, Frank copula, or No.16 copula, and \( \text{COV}_{\text{Gaussian}} \) is the probability of failure produced by the Gaussian copula. \( R_1, R_2, R_3 \) and \( R_5 \) represent the ratios calculated by \( p_f/\text{COV}_{\text{Gaussian}} \) corresponding to sliding failure mode, overturning failure mode, bearing capacity failure mode and the retaining wall system mode, respectively. It can be seen that the differences in probabilities of failure produced by different copulas are significant, especially for strong negative correlation between \( c' \) and \( \phi' \). For \( \rho = -0.75 \), the ratios \( R_1, R_2, R_3 \) and \( R_5 \) for the No.16 copula are 7427, 13553, 434, and 3065, respectively. These results indicate that both the probabilities of failure for single failure mode and retaining wall system produced by different copulas can differ significantly. Therefore, it is important to select the appropriate copulas for modeling the bivariate distribution of shear strength parameters. This problem is beyond the scope of this study. However, it should be systematically investigated in the future. Otherwise, the misuse of copula may lead to unacceptable errors in system probability of failure.

4.1.2. Discussions

It can be concluded from the aforementioned reliability results that the probabilities of failure for the retaining wall produced by the selected four copulas differ considerably, especially for strong negative correlation between \( c' \) and \( \phi' \) and small COVs of the shear strength parameters. This section will present some discussions to explain such differences in probabilities of failure produced by different copulas. First, relative locations between the limit state surfaces and the joint PDF isolines of cohesion and friction angle are presented. Second, a comparison among simulated samples of
cohesion and friction angle associated with various copulas is carried out.

To make a better comparison between the Gaussian copula and the other three copulas, Fig. 7 shows the joint PDF isolines of shear strength parameters associated with the four selected copulas for \( \rho = -0.5, \ COV_c = 0.4, \ COV_{\psi} = 0.2 \). In Fig. 7, the same PDF isoline value of 0.00025 is adopted. Mode 1, mode 2, and mode 3 represent the sliding failure mode, the overturning failure mode, and the bearing capacity failure mode, respectively. It can be observed that the joint PDF isolines using different copulas can differ significantly although the same marginal distributions and \( q / C_0 \) are used for constructing the bivariate distribution of \((c_0, \psi)\). Such differences directly result in the different probabilities of failure of the same limit state surfaces produced by different copulas. Since the joint PDF isolines associated with the Plackett and Frank copulas differ slightly, the resulting probabilities of failure also appear to be similar. On the contrary, the joint PDF isoline associated with the No.16 copula is significantly different from that associated with the Gaussian copula. Furthermore, the No.16 copula has a lower tail dependence. The resulting probabilities of failure are significantly larger than those associated with the Gaussian copula.

From the viewpoint of simulation, it is much easier to appreciate the bivariate distribution of shear strength parameters from its simulated realizations. For illustration, only the simulated samples of cohesion and friction angle from the selected four copulas for \( q / C_0 = 0.5, \ COV_c = 0.4, \ COV_{\psi} = 0.2 \) are shown in Fig. 8, in which the sample size is 1000. The contour lines of constants \( FS = 1.0, 1.3, 1.6, 2.0 \) for the retailing wall are also plotted in Fig. 8. Note that the system factor of safety \( FS \) is the union of \( FS_1 \) in Eq. (8), \( FS_2 \) in Eq. (9) and \( FS_3 \) in Eq. (14). Again, the numbers \( N \) of samples falling in the regions associated with \( FS \leq 1.0, 1.0 < FS \leq 1.3, 1.3 < FS \leq 1.6, 1.6 < FS < 2.0 \) and \( FS > 2.0 \) are shown in the corresponding regions. It can be observed that different copulas characterize different dependence structures between cohesion and friction angle. For instance, the numbers of samples falling in the

### Table 3
Comparison of probabilities of retaining wall failure among various copulas.

<table>
<thead>
<tr>
<th>( q / C_0 )</th>
<th>( \rho = -0.40 )</th>
<th>( \rho = -0.50 )</th>
<th>( \rho = -0.60 )</th>
<th>( \rho = -0.70 )</th>
<th>( \rho = -0.75 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gaussian</td>
<td>Plackett</td>
<td>Frank</td>
<td>No.16</td>
<td>Gaussian</td>
<td>Plackett</td>
</tr>
<tr>
<td>( R_1 )</td>
<td>1</td>
<td>2.5</td>
<td>13.9</td>
<td>177.9</td>
<td>46.4</td>
</tr>
<tr>
<td>( R_2 )</td>
<td>2.3</td>
<td>3.2</td>
<td>27.4</td>
<td>5.0</td>
<td>3.3</td>
</tr>
<tr>
<td>( R_3 )</td>
<td>1.4</td>
<td>1.9</td>
<td>12.1</td>
<td>4.6</td>
<td>3.2</td>
</tr>
<tr>
<td>( R_4 )</td>
<td>2.0</td>
<td>2.0</td>
<td>8.6</td>
<td>2.3</td>
<td>2.0</td>
</tr>
<tr>
<td>( R_5 )</td>
<td>2.0</td>
<td>2.0</td>
<td>8.6</td>
<td>1.7</td>
<td>1.7</td>
</tr>
<tr>
<td>( R_S )</td>
<td>2.0</td>
<td>2.0</td>
<td>8.6</td>
<td>1.7</td>
<td>1.7</td>
</tr>
</tbody>
</table>
aforementioned five regions for the Gaussian copula are 1, 40, 152, 226, and 581. They are 26, 42, 84, 217, and 631 for the No.16 copula. Compared with the Gaussian copula, more samples fall in the region associated with $FS_{1.0}$. It should be pointed out that the region associated with $FS_{1.0}$ is of more importance to engineering design because the probability of failure is basically derived from this region. Thus, geotechnical engineers often pay more attention to the difference in this region associated with different copulas. The numbers of samples falling in this region are 1, 5, 3 and 26 for the Gaussian, Plackett, Frank and No.16 copulas, respectively. These results clearly explain that the No.16 copula leads to the largest probability of failure while the Gaussian copula results in the smallest probability of failure, as shown in Fig. 5.

4.2. Example II—System reliability of rock wedge slope

The system reliability problem of rock wedge slope has been investigated by several researchers[24,30,35,39]. Unlike the series reliability of retaining wall in example I, the system reliability of rock wedge stability is a combined system reliability problem. That is, the whole system of rock wedge stability is a series reliability problem in which each failure mode is a parallel system reliability problem[35]. In these studies, the bivariate distribution of shear strength parameters is assumed to be a bivariate normal distribution. In other words, the Nataf distribution is adopted to construct the bivariate distribution of shear strength parameters. Essentially, the dependence structure between cohesion and friction angle underlying the Nataf distribution is characterized by the Gaussian copula[28,29]. This study will further explore the effect of different copulas for modeling the bivariate distribution of shear strength parameters on the system reliability of rock wedge stability.

Fig. 9 shows a tetrahedral wedge formed by two intersecting discontinuities. In Fig. 9, $H$ is the height of slope and $h$ is the height of wedge. The symbols of $\alpha$, $\Omega$ and $\epsilon$ are the inclination angles of the slope face, the upper slope surface, and the intersection line of the two discontinuity planes, respectively. $\delta_1$ and $\delta_2$ denote the dips of the discontinuity planes 1 and 2, respectively, and $\theta_1$ and $\theta_2$ are the two angles in the horizontal triangular BDC which are related to strikes of the joints[39]. Since the presence of tension cracks, external forces due to water pressure, tensioned anchors, and seismic accelerations will significantly increase the complexity of the equations for the factor of safety of the wedge[17], a wedge that is only subjected to forces due to friction, cohesion and water pressure is considered for simplicity.

To conduct system reliability analysis of a rock wedge, the relevant failure modes should be identified based on information of wedge geometry and forces acting on the wedge. These failure modes provide a basis for the formulation of limit state functions. For the tetrahedral wedge shown in Fig. 9, four different failure modes may occur as follows[15,24,39]: sliding along the line of intersection of two discontinuity planes forming the block (Failure Mode 1, also called biplane sliding); sliding along discontinuity plane 1 only (Failure Mode 2); sliding along discontinuity plane 2 only (Failure Mode 3); and a floating failure (Failure Mode 4) which could be induced by high water pressure or in-situ
Random variables in the wedge stability analysis, wedge height, inclination angles of slope face and upper ground surface, are adopted for the calculation of the factor of safety. The equations are explicit functions of discontinuity orientation, wedge height, inclination angles of slope face and upper ground surface, water pressure parameters, and friction angle and cohesion of discontinuities. Table 4 lists the physical interpretation of the failure modes and the factors of safety corresponding to each failure mode.

The following deterministic parameters are adopted for the analyses: \( h = 15 \text{ m}, \alpha = 70^{\circ}, \Omega = 0^{\circ}, \gamma_{w} = 10 \text{ kN/m}^2, \gamma_{\text{rock}} = 26 \text{ kN/m}^2, \text{ and } S_{c} = \gamma_{\text{rock}}/\gamma_{w} = 2.6 \). Random variables in the wedge stability model are assumed to be independent of each other. However, cohesion \( c' \) and friction angle \( \phi' \) of planes 1 and 2 are considered to be negatively correlated. Different copulas are used to characterize the correlation structure between cohesion and friction angle. The statistical parameters for the input variables in the wedge stability model are listed in Table 5. The random variables in Table 5 follow a normal or lognormal distribution. Like the retaining wall example, the four performance functions corresponding to the four failure modes of the rock wedge in Table 4 are expressed as \( g_{1}(X) = F_{S_{1}}(X) - 1, g_{2}(X) = F_{S_{2}}(X) - 1, g_{3}(X) = F_{S_{3}}(X) - 1, \text{ and } g_{4}(X) = F_{S_{4}}(X) - 1 \), respectively. \( F_{S_{1}}(X), F_{S_{2}}(X), F_{S_{3}}(X), \text{ and } F_{S_{4}}(X) \) correspond to the factors of safety listed in Table 4. The direct MCS with a sample size of \( m = 1 \times 10^6 \) is conducted to determine the system probability of rock wedge slope failure. This sample size is sufficient to obtain a probability of failure up to \( 1 \times 10^{-4} \), and the corresponding COV of probability of failure is smaller than 0.1.

Similar to example I, Fig. 10(a) shows the system probabilities of rock wedge slope failure associated with different copulas for \( \rho \) varying from \(-0.4 \) to \(-0.9 \). In Fig. 10(a), the system probabilities of rock wedge slope failure produced by different copulas decrease with increasing negative correlation between cohesion and friction stresses, or applied forces, or both. Note that such failure modes represent only a limited set of failure possibilities of rock wedges [16]. The closed-form equations proposed by Low [37,39] for the stability of tetrahedral wedges in rock slopes with an inclined upper ground surface, are adopted for the calculation of the factor of safety. The equations are explicit functions of discontinuity orientation, wedge height, inclination angles of slope face and upper ground surface, water pressure parameters, and friction angle and cohesion of discontinuities. Table 4 lists the physical interpretation of the failure modes and the factors of safety corresponding to each failure mode.
Table 4

<table>
<thead>
<tr>
<th>Failure mode</th>
<th>Physical interpretation</th>
<th>Factor of safety</th>
<th>Condition</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Wedge sliding on both planes</td>
<td>$F_{S1} = abG_1 \tan \phi_1 + abG_2 \tan \phi_2 + 3b_1 \frac{G_1}{C_1} + 3b_2 \frac{G_2}{C_2}$</td>
<td>$abG_1 \geq 0, abG_2 \geq 0$ (\Omega &lt; \psi &lt; \infty)</td>
</tr>
<tr>
<td>2</td>
<td>Wedge sliding on plane 1 only</td>
<td>$F_{S2} = \frac{abG_2 \tan \phi_1 + 3b_1 \frac{G_2}{C_2}}{\sqrt{1 + \left(\frac{G_1}{C_1} / \tan \phi_2\right)^2}}$</td>
<td>$abG_2 &lt; 0, abG_1 &gt; 0$ (\Omega &lt; \psi &lt; \infty)</td>
</tr>
<tr>
<td>3</td>
<td>Wedge sliding on plane 2 only</td>
<td>$F_{S3} = \frac{abG_1 \tan \phi_2 + 3b_2 \frac{G_1}{C_1}}{\sqrt{1 + \left(\frac{G_2}{C_2} / \tan \phi_1\right)^2}}$</td>
<td>$abG_1 &lt; 0, abG_2 &gt; 0$ (\Omega &lt; \psi &lt; \infty)</td>
</tr>
<tr>
<td>4</td>
<td>Wedge floating</td>
<td>$F_{S4} = 0$</td>
<td>$abG_2 &lt; 0, abG_1 &gt; 0$ (\Omega &lt; \psi &lt; \infty)</td>
</tr>
</tbody>
</table>

Note: The definitions of all symbols in Table 3 are referred to Lee et al. [30].

Table 5

Summary statistics of basic random variables in the wedge stability model.

<table>
<thead>
<tr>
<th>$\theta_1$ (°)</th>
<th>$\theta_2$ (°)</th>
<th>$\theta_3$ (°)</th>
<th>$\theta_4$ (°)</th>
<th>$G_w$</th>
<th>$\gamma_1$ (kPa)</th>
<th>$\gamma_2$ (kPa)</th>
<th>$\phi_1$ (°)</th>
<th>$\phi_2$ (°)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>62</td>
<td>40</td>
<td>50</td>
<td>48</td>
<td>0.5</td>
<td>42</td>
<td>42</td>
<td>35</td>
</tr>
<tr>
<td>COV</td>
<td>0.05</td>
<td>0.15</td>
<td>0.04</td>
<td>0.04</td>
<td>0.24</td>
<td>0.30</td>
<td>0.30</td>
<td>0.20</td>
</tr>
<tr>
<td>Distribution</td>
<td>Normal</td>
<td>Normal</td>
<td>Normal</td>
<td>Normal</td>
<td>Normal</td>
<td>Lognormal</td>
<td>Lognormal</td>
<td>Lognormal</td>
</tr>
</tbody>
</table>

Fig. 10. System probabilities of rock wedge slope failure associated with different copulas.

angle. Different copulas will lead to different system probabilities of failure. The system probabilities of failure produced by the No.16 copula are significantly larger than those produced by the other three copulas. In contrast, the often used Gaussian copula results in the smallest system probabilities of failure, which may be unconservative for safety assessment of geotechnical structures. For $\rho = -0.7$, the system probability of failure associated with the No.16 copula is about 20 times larger than that associated with the Gaussian copula. Similarly, Fig. 10(b) shows the system probabilities of rock wedge slope failure associated with different copulas for $\lambda$ varying from 0.8 to 1.5. Similar observations as those obtained from Fig. 10(a) can also be made. These results provide a new insight on the frequently used assumption that the Gaussian copula is adequate for characterizing the correlation structure between cohesion and friction angle.

5. Summary and conclusions

This paper has investigated the effect of copulas for constructing bivariate distribution of shear strength parameters on system reliability of geotechnical structures under incomplete probability information. Two illustrative examples are presented to demonstrate the effect of copula selection on system reliability. Several conclusions can be drawn from this study:

1. Copulas provide an effective tool for constructing the joint distribution of correlated geotechnical parameters under incomplete probability information. The joint PDFs of cohesion and friction angle under incomplete probability information are not determined uniquely. Consequently, the system reliability of geotechnical structures under incomplete probability information cannot be determined uniquely. This finding should be noted in system reliability evaluation of geotechnical structures.

2. The copulas for modeling the dependence structure between shear strength parameters have a significant influence on the system reliability of geotechnical structures because different copulas produce different bivariate distribution of shear strength parameters. Such an influence includes two phases separately. The first phase is that the dependence
structure between shear strength parameters characterized by different copulas affects the reliability of single failure mode, depending on the marginal distributions, dependence structure of shear strength parameters, and reliability level of each failure mode. The second phase is that the reliability of each failure mode influences on system reliability of geotechnical structures, only depending on reliability level of each failure mode and correlations among various failure modes.

(3) It is important to distinguish between the effect of copula selection on reliability of each failure mode and that on the system reliability of the geotechnical structures. The selection of an appropriate copula for modeling the dependence structure between shear strength parameters is still a challenging problem and should be further investigated in the future.

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