A system reliability approach for evaluating stability of rock wedges with correlated failure modes

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**ABSTRACT**

This paper proposes a system reliability approach for evaluating the stabilities of rock wedges considering multiple correlated failure modes. A probabilistic fault tree is employed to model the system aspects of the problem. The system reliability analysis is performed using an N-dimensional equivalent method taking into account correlations between different failure modes. Reliability sensitivity analyses at three different levels, namely, single limit state function level, single failure mode level, and system reliability level, were carried out to study the effect of changes in variables on the stability of the wedge. An example case was analysed to illustrate the proposed approach. The stability of the wedge can be evaluated efficiently using the proposed system reliability approach in a more systematic and quantitative way. The probabilities of failure of the wedge from the N-dimensional equivalent method are fairly consistent with those from the Monte Carlo simulation method. The results demonstrate that the probability of failure will be overestimated if the correlations between different failure modes of the wedge are not taken into account. They also demonstrate that the relative importance of different failure modes to the system reliability of the wedge can differ considerably and be treated systematically and quantitatively by the proposed approach. The sensitivity results are highly dependent on the selected sensitivity analysis level.

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1. Introduction

Wedge failures are the most frequently observed rock slope failures, and can occur over a wide range of geological and geometrical conditions. Accordingly, the study of wedge stability is an important component for rock slope engineering. Wedge failures are often governed by the intersection of at least two rock discontinuity sets, which require a solution of forces in three-dimensional space [20,15]. The mechanisms leading to wedge failures in rock slopes have been extensively studied in the literature [17,31,10,11,25,30]. The methods used include a stereographic projection technique, engineering graphics, and vector analysis. Also, Low [21] proposed compact closed-form equations for the factor of safety of two-joint tetrahedral wedges. Most of these analyses are based on deterministic methods which do not reflect the uncertainty of the underlying parameters. It is widely accepted that rock wedge stability analysis often contains many uncertainties due to inadequate information from site characterization, and inherent variability and measurement errors in the geological and geotechnical parameters. Therefore, reliability-based approaches which allow the systematic and quantitative treatment of these uncertainties have become a topic of increasing interest for rock slope engineering [19,6,23].

In the literature, Low [21] proposed closed-form equations for the calculations of the factor of safety for the wedge stability in rock slopes with an inclined upper ground surface that dips in the same direction as the slope face. The system reliability of the wedge considering four failure modes is evaluated using Cornell’s bound method [4] and Monte Carlo simulation. Low [22] further investigated the system reliability of the wedge in which the versatile four-parameter beta distributions are used for describing the basic random variables in the rock wedge stability model. Based on Low [21], Jimenez-Rodriguez and Sitar [18] explored a disjoint cut-set formulation to model the system reliability of the wedge in which each cut-set corresponds to a failure mode of the wedge. Fadlelmula et al. [8] compared the failure probabilities of a wedge when the Coulomb linear shear failure criterion and the Barton–Bandis non-linear shear failure criterion were applied. However, the disjoint cut-set formulation cannot account for the correlations between different failure modes of the wedge. Although such correlations may be taken into account in Monte Carlo simulations, it is too time consuming to be of practical interest to engineers, and the sensitivity analysis at different reliability levels has not been investigated sufficiently.

The objective of this paper is to propose a system reliability approach for evaluating the stability of rock wedges with multiple
correlated failure modes. First, limit state functions for wedge failures are formulated. Then the system aspects of the wedge stability analysis using limit equilibrium methods are represented by a probabilistic fault tree [2]. The versatile four-parameter beta distributions are used to describe the basic random variables defined in the wedge stability model. An N-dimensional equivalent method is proposed to perform the reliability analysis of the wedge. Finally, an example is presented to illustrate the proposed method. The importance of considering correlations between different failure modes and the advantages of system reliability analysis over the traditional deterministic approaches are also demonstrated. Reliability sensitivity analyses at three different levels, namely, single limit state function level, single failure mode level, and system reliability level, are carried out to evaluate the effect of changes in variables on the stability of the wedges.

2. Formulation of limit state function for wedge failure

The first step in the system reliability analysis of a wedge is to identify the relevant failure modes based on information of wedge geometry and forces acting on the wedge. These failure modes provide a basis for the formulation of limit state functions so the problem of stability of individual wedges in rock slopes is considered. A tetrahedral wedge may be formed by two intersecting discontinuities (Fig. 1), with a typical wedge geometry where $H$ is the height of slope and $h$ is the height of wedge. The symbols of $x$, $d$, and $e$ are the inclination angles of the slope face, the upper slope surface, and the intersection line of the two discontinuity planes, respectively. $\phi_1$ and $\phi_2$ denote the dips of the discontinuity planes 1 and 2, respectively, and $\phi_1$ and $\phi_2$ are the two angles in the horizontal triangular BDC which are related to strikes of the joints [21]. Since the presence of tension cracks, external forces due to water pressure, tensioned anchors, and seismic accelerations will significantly increase the complexity of the equations for the factor of safety [15], a wedge that is only subjected to forces due to friction, cohesion and water pressure is considered for simplicity.

For the tetrahedral wedge shown in Fig. 1, four different failure modes may occur as follows [10,21]: sliding along the line of intersection of two discontinuity planes forming the block (Failure Mode 1, also called biplane sliding); sliding along discontinuity plane 1 only (Failure Mode 2); sliding along discontinuity plane 2 only (Failure Mode 3); and a floating failure (Failure Mode 4) which could be induced by high water pressure or in situ stresses, or applied forces, or both. Note that such failure modes represent only a limited set of failure possibilities of rock wedges [12].

With the identified failure modes, limit state functions for the wedge can be formulated as follows: let $X$ denote a vector of all random variables that should be taken into account to evaluate the wedge stability. The random variables include orientations, cohesions and friction angles of the discontinuities, and the loading conditions. The limit state function for the wedge stability can be expressed as:

$$g(X) = F_S - 1$$

where $F_S$ is the factor of safety. If $g(X)$ is less than zero, the wedge is in the failure domain. Otherwise it is in the safe domain. Function $g(X) = 0$ represents the limit state surface. The formulations of $F_S$ are discussed below.

The closed-form equations proposed by Low and Einstein [24], Low [21], and Low [22] for the stability of tetrahedral wedges in rock slopes with an inclined upper ground surface are adopted for the calculation of the factor of safety. The equations are explicit functions of discontinuity orientation, wedge height, inclination angles of slope face and upper ground surface, water pressure parameters, and friction angle and cohesion of discontinuities.

The factor of safety for a biplane sliding mode is:

$$F_S = \left( \frac{a_1 - b_1 G_{w1}}{s_1} \right) \times \tan \phi_1 + \left( \frac{a_2 - b_2 G_{w2}}{s_2} \right) \times \tan \phi_2$$

$$+ 3b_1 c_1 \frac{1}{\gamma} + 3b_2 c_2 \frac{1}{\gamma}$$

$$F_S = (3b_1 c_1 \gamma + 3b_2 c_2 \gamma)$$

where $c_1, c_2, \phi_1, \phi_2$ are the cohesions and friction angles of discontinuities 1 and 2; $G_{w1}$ and $G_{w2}$ are the water pressure parameters; $\gamma$ is the unit weight of rock; $s = \gamma/\gamma_w$ is the specific density of rock where $\gamma_w$ is the unit weight of water; $a_1, a_2, b_1, b_2$ are parameters depending on the geometry of the wedge, which can be calculated by:

$$a_1 = \frac{\sin \phi_3 \cos \phi_1 - \cos \phi_3 \cos (\phi_1 + \phi_2)}{\sin \phi_1 (\phi_1 + \phi_2)}$$

$$a_2 = \frac{\sin \phi_3 \cos \phi_2 - \cos \phi_3 \cos (\phi_1 + \phi_2)}{\sin \phi_1 (\phi_1 + \phi_2)}$$

$$b_1 = a_0 \sin \phi_1 \sin \phi_2$$

$$b_2 = a_0 \sin \phi_1 \sin \phi_2$$

with:

$$\sin \psi = \sqrt{1 - \left[ \sin \phi_1 \sin \phi_2 \cos (\phi_1 + \phi_2) + \cos \phi_1 \cos \phi_2 \right]^2}$$

$$a_0 = \frac{\sin \psi}{\sin (\phi_1 + \phi_2) \sin \phi_1 \sin \phi_2 (\cot \varepsilon - \cot \phi)}$$

$$\varepsilon = \arctan \left( \frac{\sin \phi_1 \cot \phi_2 + \sin \phi_2 \cot \phi_1}{\sin \phi_1 \cot \phi_2 + \sin \phi_2 \cot \phi_1} \right)$$

For a pyramidal pressure distribution as shown in Fig. 2 [15], one can obtain:

$$G_{w1} = G_{w2} = 0.5 \kappa$$

$$\kappa = \frac{H}{h} = \frac{1 - \tan \Omega}{\tan \gamma} \left( \frac{1 - \tan \Omega}{\tan \varepsilon} \right)$$

where $h$ and $H$ become equal when $\Omega$ is equal to zero. If only the length of $DC$, as shown in Fig. 1, is known $h$ can be obtained from:

$$h = \frac{DC (\cot \varepsilon - \cot \phi)}{(\cot \phi_1 + \cot \phi_2)}$$

Note that Eq. (2) is valid when the following conditions are met:
Substituting Eq. (2) into Eq. (1), the limit state function can be obtained as:

\[
g(X) = \left( a_1 - \frac{b_1 C_{w1}}{S_1} \right) \times \tan \phi_1 + \left( a_2 - \frac{b_2 C_{w2}}{S_2} \right) \times \tan \phi_2 + 3b_1 \frac{C_1}{\gamma H} + 3b_2 \frac{C_2}{\gamma H} - 1
\]  

(14)

The basic random variables in this limit state function are \( C_1, C_2, \) \( C_{w1}, C_{w2}, \phi_1, \phi_2, \theta_1, \sigma \) and \( \sigma_2, \) which are the same as those for all the other failure modes. Symbols \( \alpha, \Omega, \gamma, \) and \( \sigma_j \) in Eq. (13) are deterministic physical parameters.

When Eq. (13) is not valid, several failure modes (sliding along discontinuity plane 1 only, sliding along discontinuity plane 2 only, or floating failure) should be considered. The factor of safety, \( F_{S1}, \) for sliding along discontinuity plane 1 only is given by:

\[
F_{S1} = \left( a_1 - \frac{b_1 C_{w1}}{S_1} \right) - \left( a_2 - \frac{b_2 C_{w2}}{S_2} \right) \tan \phi_1 + 3b_1 \frac{C_1}{\gamma H} \left[ 1 + \left( \frac{b_1 C_{w1}}{S_1} - a_1 \right) \sin \psi \right]^2
\]  

(15)

where

\[
Z = \cos \theta_1 \cos \phi_2 + \sin \theta_1 \sin \phi_2 \cos (\phi_1 + \theta_2)
\]  

(16)

All other symbols in Eqs. (15) and (16) are as defined previously. Eq. (15) is valid only when the following conditions are satisfied:

\[
\begin{align*}
& \left( a_2 - \frac{b_2 C_{w2}}{S_2} \right) < 0 \\
& \left( \left( a_1 - \frac{b_1 C_{w1}}{S_1} \right) - \left( \frac{b_2 C_{w2}}{S_2} - a_2 \right) \right) > 0
\end{align*}
\]  

(17)

Substituting Eq. (15) into Eq. (1), the limit state function can be obtained:

\[
g(X) = \left( a_1 - \frac{b_1 C_{w1}}{S_1} \right) - \left( a_2 - \frac{b_2 C_{w2}}{S_2} \right) Z \tan \phi_1 + 3b_1 \frac{C_1}{\gamma H} \left[ 1 + \left( \frac{b_1 C_{w1}}{S_1} - a_1 \right) \sin \psi \right]^2
\]  

(18)

Similarly, the factor of safety, \( F_{S2}, \) for sliding along discontinuity plane 2 only is obtained as:

\[
F_{S2} = \frac{\left( a_1 - \frac{b_1 C_{w1}}{S_1} \right) - \left( \frac{b_2 C_{w2}}{S_2} - a_2 \right) Z \tan \phi_2 + 3b_2 \frac{C_2}{\gamma H}}{\left[ 1 + \left( \frac{b_1 C_{w1}}{S_1} - a_1 \right) \sin \psi \right]^2}
\]  

(19)

where all symbols are as defined previously. Eq. (19) is valid only when:

\[
\begin{align*}
& \left( a_1 - \frac{b_1 C_{w1}}{S_1} \right) < 0 \\
& \left( a_2 - \frac{b_2 C_{w2}}{S_2} - \left( \frac{b_2 C_{w2}}{S_2} - a_2 \right) \right) > 0
\end{align*}
\]  

(20)

Floating failure could be induced by high water pressures or in situ stresses as previously discussed. This scenario occurs when the following conditions are met:

\[
\begin{align*}
& \left( a_1 - \frac{b_1 C_{w1}}{S_1} \right) - \left( \frac{b_2 C_{w2}}{S_2} - a_2 \right) Z < 0 \\
& \left( a_2 - \frac{b_2 C_{w2}}{S_2} - \left( \frac{b_2 C_{w2}}{S_2} - a_2 \right) \right) > 0
\end{align*}
\]  

(21)

In addition to these failure modes, the kinematical constraint for the formation of a tetrahedral wedge must be fulfilled, and is given by the following inequality:

\[
\Omega < \zeta < \alpha
\]  

(23)

If the condition represented by inequality (23) is not satisfied, a tetrahedral wedge mechanism is not possible and the slope is safe as far as the wedge failure is concerned. It should be noted that the Eqs. (1)–(23) were initially derived by Low and Einstein [24], Low [21], and Low [22], and are used in this study for the system reliability analysis of the wedge.

3. System reliability approach

Many physical systems that are composed of multiple components can be classified as either series systems, parallel systems, or combined systems. A system of single components is a series system if it is in a state of failure whenever any one of its components fail. Such a system is also called a weakest link system. In other words, the reliability or safety of the system requires that none of the components fail. A system of single components is a parallel system if it is in a state of failure when all of its components fail. In other words, if any one of the components survives, the system remains safe. Many structures can be considered as a combination of series and parallel systems. Such systems are referred to as combined systems. More detailed information can be found in Ang and Tang [1]. In the most general case in the field of system reliability evaluation, the probability of failure of a structural system can be modelled by a series of parallel systems such as failure modes. This combined system may be defined in the form:

\[
P_I = P \left[ \bigcup_{k = 1} \bigcap_{C_k} g_j(X) \leq 0 \right]
\]  

(24)

where \( g_j(X) \) is the jth limit state function; \( C_k \) denotes the kth subset representing a set of limit states whose joint exceed constitutes the failure of the system, and the union is over all of the subsets. In Eq. (24), \( C_k \) contains a single element for each subset, k, for the special class of a series system, and a single subset, \( k = 1, \) for the special class of a parallel system.

The reliability of the above general structural system is evaluated by the following steps: First, the probability of failure of each parallel system is evaluated. Secondly, the evaluation of the correlations between the parallel systems due to common variables or correlated variables is performed. Finally, the probability of failure for the series system is evaluated on the basis of the results.
obtained from the first two steps. Evaluation of the correlation between a pair of parallel systems can be easily carried out if the safety margins for the parallel systems are linear. However, in general this is not the case. Therefore an alternative is to investigate the possibility of introducing an equivalent linear safety margin for each parallel system, which will be discussed later. Jimenez-Rodriguez et al. [19], and Jimenez-Rodriguez and Sitar [18] used a disjoint subset formulation in which the performance of the system is modelled as a series assembly of disjointed parallel sub-systems. The total probability of failure of the system may be obtained as the sum of the individual failure modes. That is, the correlations between different failure modes are not considered, which leads to an overestimation of the system probability of failure for the wedge. To evaluate the correlations between different failure modes efficiently, a probabilistic fault tree is developed to solve the general system problem as described below.

3.1. Probabilistic fault tree analysis

In the field of system reliability evaluation, fault tree analysis (FTA) provides an organized means for identifying sources of structural system failure and their interactions that may lead to one or more failure paths. The fault tree provides a risk assessment tool by which a complicated structural system can be managed systematically and quantitatively. It is not practical to present detailed discussion on how to identify failure events and paths in this paper and interested readers can consult Ang and Tang [1] for more information. We focus here on how these failure paths are modelled using a probabilistic fault tree [2], which provides a systematic way to manage multiple failure modes.

A probabilistic fault tree has three major characteristics: bottom events, combination gates, and the connectivity between the bottom events and gates. Only AND and OR gates are currently included in the probabilistic fault tree approach. The AND gate is used to model a parallel system while the OR gate is used to model a series system. The limit state functions are defined in the bottom events so that the correlations between different failure modes represented by the limit state functions can be considered. In a conventional fault tree approach, however, probability values are first assigned to all bottom events that are assumed to be independent. Next, they are propagated through the logic gates of the fault tree to calculate the probability of failure. Through a probabilistic fault tree analysis, all the failure modes can be defined, and the correlations between different failure modes can be taken into consideration in a more rational way. Note that a failure mode can involve one or more limit states. By combining all the failure modes and the corresponding limit states, a system limit state surface can be constructed piece by piece.

Based on the four failure modes of a wedge, the system reliability of the wedge can be modelled by a probabilistic fault tree as shown in Fig. 3. Note that the performance of the system is modelled as a series of all parallel systems (failure modes). That is, the failure of the overall system (wedge) will occur when any failure mode occurs. For each failure mode involving several components, failure will occur only when all components in the corresponding parallel subsystem fail. The performance of each component is defined by a limit state function. Table 1 lists the physical interpretation of the limit state functions and the definitions of the limit state functions corresponding to each failure mode.

Having constructed a probabilistic fault tree for a system, the reliability of the system can be calculated using the system reliability approach. For reliability evaluation of a system, the reliability of each component should be computed first as a probability of failure $P_i$, given by:

$$P_i = P(g_i(X) \leq 0) = \int_{g_i(X) \leq 0} f(X) dX$$

where $f(X)$ is the probability density function of the variables associated with the rock wedge stability model. The probability of failure in Eq. (25) can be easily computed by the component reliability methods, such as the FORM [13,26], the SORM [3] and the Monte Carlo simulation method (e.g., [1]). The system reliability modelled by the probabilistic fault tree can be calculated using different system reliability evaluation methods, such as the narrow reliability bounds method [5], the adaptive importance sampling method [32], and the first order multinormal (FOMN) method proposed by Hohenbichler and Rackwitz [16]. The N-dimensional equivalent method is, however, used to perform the system reliability analysis due to its accuracy and efficiency as demonstrated below.

3.2. N-dimensional equivalent method

Consider the case where a series system is composed of $n$ components. A useful first step is to transform all safety margins associated with the $n$ components to their standardized forms using the FORM as follows:

$$Z_i = -\frac{x_i}{\sigma_i} Y_i + \beta_i \quad i = 1, 2, \ldots, n$$

$\sigma_i$, $Y_i$, and $\beta_i$ are the standard deviation, mean, and reliability of the failure surface, respectively. Fig. 3. Probabilistic fault tree model for system reliability of wedge stability.
Let the vector \( \vec{Y} \) of the basic variables be increased by an increment represented by vector \( \Delta \vec{Y} \). Then the corresponding probability of failure for the series system is:

\[
P_{f}(\Delta \vec{Y}) = 1 - \sum_{i=1}^{n} p_{i} Y_{i} + \Delta Y_{i} + \beta_{i} \geq 0
\]

where \( [\Delta \vec{Y}] \) is the matrix consisting of the vector of direction cosines for each failure surface. Based on Eq. (32), the corresponding nominal reliability index increment for \( \vec{Y} \) with an increment vector \( \Delta \vec{Y} \) can be given by:

\[
\beta_{k}(\Delta \vec{Y}) = -\Phi^{-1}\left[ p_{k}(\Delta \vec{Y}) \right] = -\Phi^{-1}\left[ 1 - \Phi_{n}(\beta - [\Delta \vec{Y}] ; \rho) \right]
\]

The safety margin of equivalent failure surface for \( \vec{Y} \) with an increase of \( \Delta \vec{Y} \) is:

\[
Z^{e}(\Delta \vec{Y}) = -\vec{z}^{eT} \vec{Y} - \vec{z}^{e} \Delta \vec{Y} + \beta^{e}
\]

and the corresponding equivalent reliability index is:

\[
\beta^{e}(\Delta \vec{Y}) = -z_{k}^{e} \Delta Y_{k} - z_{k}^{e} \Delta Y_{k} - \ldots - z_{m}^{e} \Delta Y_{m} + \beta^{e}
\]

where \( z_{k}^{e} \) is equal to the derivative of \( \beta^{e}(\Delta \vec{Y}) \) with respect to \( Y_{k} \), expressed as:

\[
z_{k}^{e} = -\frac{\partial \beta^{e}(\Delta \vec{Y})}{\partial Y_{k}}\bigg|_{\Delta Y_{k}=0} = -\frac{1}{l} \frac{\partial p_{k}(\Delta \vec{Y})}{\partial Y_{k}}\bigg|_{\Delta Y_{k}=0}
\]

The equivalent linear safety margin representing a parallel system can also be constructed using a similar method. In general, the probability of failure for a parallel series system is:

\[
P_{f}(\vec{Y}) = 1 - \Phi_{n}(\beta ; \rho)
\]

The equivalent linear safety margin for a parallel system can be determined by replacing Eqs. (28), (32), and (33) with the following equations, respectively:

\[
\beta_{k}(\vec{Y}) = -\Phi^{-1}\left[ p_{k}(\vec{Y}) \right]
\]

\[
P_{f}(\vec{Y}) = 1 - \sum_{i=1}^{n} p_{i} Y_{i} + \Delta Y_{i} + \beta_{i} \geq 0
\]

\[
\beta_{k}(\vec{Y}) = -\Phi^{-1}\left[ p_{k}(\vec{Y}) \right]
\]

The remainder of the derivation steps are identical with those for a series system.

After obtaining the equivalent linear safety margins using the above method, the correlation coefficients, \( \rho_{ij} \) between the equivalent safety margins can be given by:

\[
\rho_{ij} = \frac{\vec{z}_{i} \cdot \vec{z}_{j}}{\vec{z}_{i} \cdot \vec{z}_{i}}
\]
3.3. Numerical implementation for reliability computation

In this study, a C-language based computer program WHUREL (Wuhan University Reliability computer program for rock slopes) was developed for calculating the reliability index \( \beta \) and the most probable failure points \( X^* \). WHUREL is capable of calculating the reliability of a component using FORM as well as the system reliability of a series system or a parallel system, using the proposed \( N \)-dimensional equivalent method. The Cornell’s bound method and the narrow reliability bounds method [5], for reliability analysis of a series system, are also included in WHUREL. In addition, WHUREL can perform both the component reliability sensitivity analyses and system reliability sensitivity analyses for the basic random variables. A major advantage of WHUREL is that the limit state functions, defined by the bottom events in the probabilistic fault tree as shown in Fig. 2, can be expressed in the form of a set of user-defined subroutines.

4. An illustrative example

4.1. Case geometry and material properties

As an example, the rock wedge stability model shown in Fig. 1 is investigated. The following deterministic parameters are adopted for the analyses: \( \alpha = 70 ^\circ \), \( \Omega = 0 \), \( \gamma_w = 9.8 \text{ (kN/m}^2) \), and \( s_r = 2.6 \) [21,18]. The height of wedge, \( h \), ranging from 10 to 30 m, is used to account for the effect of \( h \) on the system reliability of the wedge. Random variables in the wedge stability model are assumed to be independent of each other. However, cohesion \( c \) and friction angle \( \phi \) are considered to be negatively correlated. Correlation coefficients of \( \rho_{c1,c2} = \rho_{c2,c3} = -0.5 \) are used to model common shear test results in which the cohesion generally decreases as the friction angle increases and vice versa [22,23,14]. Note that for rock discontinuities, with the development of shear, the cohesion will become zero and this will not be reversible, i.e. this loss of cohesion cannot be recovered. Therefore, in a forward and reversed shear path, the cohesion should not be recoverable. This may not be applicable for a wedge failure, but one may need to consider such possibilities in other applications. The statistical parameters for the input variables in the wedge stability model are listed in Table 2. All variables in Table 2 follow a beta distribution.

It should be noted that, instead of the normal distributions or lognormal distributions used in Low [21], Duzgun et al. [7], and Jimenez-Rodriguez and Sitar [18], all variables are modelled by the four-parameter \( (q, r, a, b) \) beta distributions in which the first two parameters are shape parameters, while the last two parameters define the lower and upper limits of the range. This is because the normal distribution is symmetrical and, theoretically, has a range from \(-\infty \) to \(+\infty \). For a parameter that admits only positive values, the probability of encroaching into the negative realm is very small if the coefficient of variation of the parameter is 0.25 or less [22]. Since the lognormal distribution excludes negative values and facilitates the mathematical derivations, it has been suggested in lieu of the normal distribution. The lognormal distribution is within the range of \((0, +\infty)\). Both normal distributions and lognormal distributions are not suitable because all the considered variables in the wedge stability model admit only positive values, and are bounded. Compared with the normal or lognormal distributions, the four-parameter beta distribution is more versatile as demonstrated by Low [22], Low [23], which can be symmetrical if \( q = r \) or nonsymmetrical if \( q \neq r \). The mean, \( \mu_x \), and standard deviation, \( \sigma_x \), of random variable \( X \) following a beta distribution are given by (e.g., [1]):

\[
\mu_x = a + \frac{q}{q+r} (b-a) \\
\sigma_x = \sqrt{\frac{qr}{(q+r)^2(q+r+1)}} (b-a)^2
\]

In Table 2, it is assumed that \( c_1 = c_2 \), and \( \phi_1 = \phi_2 \). These assumptions, while not essential, are adopted for simplicity. The proposed method can handle much more complicated scenarios.

4.2. Analysis results

WHUREL is employed to perform the reliability analyses of the considered wedge stability. To illustrate the computations of the system reliability of the wedge using the \( N \)-dimensional equivalent method, we first take the wedge with a height of 20 m. As can be seen from Fig. 3, the system of the wedge stability is composed of four failure modes with a series-connected relationship between them. Among them, the Failure Mode 1 consists of five elements with a parallel-connected relationship between them. The reliability indices for single elements are calculated using the FORM, which are 1.613, -4.298, -1.388, -4.298, and -4.300 for \( g_1 \leq 0 \), \(-g_2 \leq 0 \), \(-g_3 \leq 0 \), \( g_4 \leq 0 \), and \( g_5 \leq 0 \), respectively. The correlation matrix can be determined using Eq. (42):

\[
[r] = \begin{bmatrix}
1 & -0.133 & -0.654 & 0.212 & -0.179 \\
-0.133 & 1 & -0.111 & -0.000149 & 0.000913 \\
-0.654 & -0.111 & 1 & -0.343 & 0.535 \\
0.212 & -0.000149 & -0.343 & 1 & -0.000428 \\
-0.179 & 0.000913 & 0.535 & -0.000428 & 1
\end{bmatrix}
\]

Then, the reliability index for the Failure Mode 1 can be calculated as 1.905 using Eq. (39), and the corresponding probability of failure is 0.0284 using Eq. (38). Applying the \( N \)-dimensional equivalent method, the equivalent linear safety margin \( Z_1 \) for the parallel system corresponding to the Failure Mode 1 can be given by:

\[
Z_1 = 0.0807c_1 - 0.628c_2 - 0.36\phi_1 - 0.428\phi_2 + 0.215\delta_1 -0.0215\delta_2 - 0.165\delta_1 - 0.106\delta_2 + 0.447G_{nw} + 1.905
\]

Similarly, the reliability indices for the Failure Modes 2–4 are 1.388, 4.362, and 5.563, respectively. In addition, the equivalent linear safety margins \( Z_2 \)–\( Z_4 \) for the parallel systems corresponding to the Failure Modes 2–4 as shown in Fig. 3 can be obtained. Based on the equivalent linear safety margins \( Z_1 \)–\( Z_4 \), the correlation matrix for \( Z_1 \)–\( Z_4 \) can be calculated using Eq. (42) as follows:

\[
[r] = \begin{bmatrix}
1 & 0.355 & 0.269 & 0.0214 \\
0.355 & 1 & -0.0694 & -0.535 \\
0.269 & -0.0694 & 1 & -0.0208 \\
0.0214 & -0.535 & -0.0208 & 1
\end{bmatrix}
\]
The system reliability index of the wedge is 1.263 using Eq. (28), and the corresponding system probability of failure is 0.0103 using Eq. (27). The corresponding equivalent linear safety margin for the series system consisting of the above four failure modes as shown in Fig. 3 can be expressed as:

\[ Z = -0.0245c_1 + 0.190c_2 + 0.109\phi_1 + 0.129\phi_2 + 0.0296\delta_1 - 0.449\delta_2 + 0.119r_1 + 0.324r_2 - 0.783G_{aw} + 1.263 \]  

To conduct the verification and validation for the N-dimensional equivalent method, the results obtained from the N-dimensional equivalent method are compared with those obtained from the Monte Carlo simulation method. For simplicity and illustrative purpose, all the variables in Table 2 are tentatively assumed to be lognormal distributions with the means and standard deviations as shown in the table. Moreover, the correlation coefficients of \( \rho_{1,2,3} = \rho_{2,3,1} = -0.5 \) are used again. Table 3 shows the system reliability between the Monte Carlo simulation method and the N-dimensional equivalent method. Note that fairly good agreement was obtained between the results using the N-dimensional equivalent method and the exact solutions using the Monte Carlo simulation method when the wedge height was varied from 10 m to 30 m. For instance, the resulting relative errors in the system reliability index and the system probability of failure are smaller than 4% and 2%, respectively. Thus, the N-dimensional equivalent method can be used to compute system reliability efficiently and accurately.

Applying the procedure demonstrated earlier, the reliability indices and the corresponding probabilities of failure for various wedge heights can be determined. To reflect the contribution of each failure mode to the system probability of failure for the wedge, the probabilities of failure on a log scale for each failure mode to the system probability of failure for the wedge, while the wedge height has almost no influence on the probability of failure for Failure Modes 2 and 4. For example, when the wedge height varies from 10 to 30 m, the probability of failure for Failure Mode 1 increases from 1.748 \( \times 10^{-8} \) to 0.37. It can be seen that the contribution of each failure mode to the system probability of failure depends on the geometry of the wedge, and the cohesion and friction angle of the discontinuities forming the wedge. It should be noted, however, that when the height of the rock wedge increases, the areas of planes 1 and 2 also increase. Based on the concepts of spatial variation [29], we should, in theory, use the reduced variances for variables such as cohesion and friction angle. For simplicity, the spatial variation is not taken into consideration in the present study. However, this should be investigated further so that the comparison between system reliability for various wedge heights can be conducted in a more rational way.

As indicated earlier, Failure Modes 1 and 2 are the failure modes contributing most to the overall probability of failure. Therefore, the correlation between Failure Modes 1 and 2 should be investigated. Fig. 5 shows the correlation coefficient between Failure Modes 1 and 2 as a function of the wedge height. It can be seen that the correlation coefficients vary from 0.257 to 0.452 with increasing wedge height, which implies that Failure Modes 1 and 2 are correlated failure modes rather than independent failure modes. This correlation will significantly affect the stability of the wedge and should be properly taken into account. Neglecting the correlations between different failure modes of the wedge will lead to an overestimated system probability of failure as demonstrated in Fig. 6. In addition, due to the increase in the probabilities of failure associated with Failure Modes 1 and 2 with increasing wedge height, the correlation between Failure Modes 1 and 2 becomes stronger.

To compare the system probabilities of the failure of the wedge using different methods, Fig. 6 shows the system probabilities of the failure of the wedge as a function of wedge height using Cornell's bound method [4], Jimenez-Rodriguez and Sitar's method [18], and the N-dimensional equivalent method. Note that the system probability bounds using Cornell's bound method are too wide to be of any practical interest. However, when the probabilities of failure for all failure modes are very small for small wedge heights, the system probability bounds become much narrower, and hence sufficiently accurate. The system probability bounds using the proposed N-dimensional equivalent method are fully within the computed Cornell's bounds, which also indicate that the proposed N-
the system probabilities of failure using the relatively small. For example, for a wedge with a height of 15 m, the system probabilities of failure using the perennial method, the Jimenez-Rodriguez and Sitar’s method, and Cornell’s bound method are 0.0829, 0.0832, and (0.0825, 0.0831), respectively. As expected, the system probabilities of failure appear to be similar.

Comparison between system probabilities of failure using different methods.

Fig. 5. Correlation coefficient between Failure Modes 1 and 2 as a function of wedge height.

Fig. 6. Comparison between system probabilities of failure using different methods.

dimensional equivalent method is valid. On the other hand, the system probability bounds using the Jimenez-Rodriguez and Sitar’s method are higher than the Cornell upper bounds because Jimenez-Rodriguez and Sitar [18] assumed that the system probability of failure for the wedge can be taken as the sum of probabilities of failure associated with the considered four failure modes. Accordingly, the system probability of failure of the wedge will be overestimated, especially for larger wedge heights. For the wedge with a height of 30 m, the system probabilities of failure using the proposed method and the Jimenez-Rodriguez and Sitar’s method are 0.385 and 0.453, respectively. It should be noted, however, that the difference in the system probability of failure among the above three methods becomes very small when the wedge height is relatively small. For example, for a wedge with a height of 15 m, the system probabilities of failure using the N-dimensional equivalent method, the Jimenez-Rodriguez and Sitar’s method, and Cornell’s bound method are 0.0829, 0.0832, and (0.0825, 0.0831), respectively. As expected, the system probabilities of failure appear to be similar.

In practice, the negative correlation between cohesion and friction angle is often not modelled, to simplify computations. To account for the effect of negative correlation between the cohesion and friction angle, the variation of reliability index with the correlation coefficient ρ between c and φ is shown in Fig. 7. Note that the system reliability index slightly increases with ρ. For instance, when ρ varies from 0 to −0.99, the system reliability index only increases from 1.206 to 1.308. The system reliability of the wedge will be underestimated if the correlation between c and φ is not taken into consideration.

4.3. Sensitivity analysis

An analysis is performed to assess the sensitivity of the random variables to the reliability of the wedge for each failure mode as well as to the overall system performance. Such a sensitivity analysis could be useful in cost analysis and design planning. For example, if the sensitivity of a variable is low there is little need to be very accurate about the determination of this variable. Also, if necessary, the variable might well be treated as a deterministic rather than a random variable, which will reduce the dimensionality of the space of random variables. Such analyses are carried out using WHUREL at three levels, namely, the single limit state function level, the single failure mode level, and the system reliability level, which are discussed below. When the basic random variables are not independent, the sensitivity coefficients defined in this study are not informative in relation to the basic random variables because of the transformation to independent standardized space. For this reason, the cohesion and friction angle are assumed to be independent, even though the statistics of basic random variables with beta distributions in Table 2 are used again. It should be noted that all sensitivity coefficients defined and computed in this paper are dimensionless.

Table 4 shows the FORM results for the case of single limit state functions associated with the wedge stability. The results include the sensitivity coefficients α’ representing the sensitivity of the computed reliability results to changes in the random variables as defined in FORM, together with the computed design points X’ corresponding to the most likely failure point transformed back to the original space. A positive sign is often used when the corresponding basic variable is a load variable, while a negative sign is often used when it is a resistance variable [28]. Note that the larger the absolute value of the ith component of α’ corresponding to random variable X_i the higher is the sensitivity with respect to the ith random variable X_i.

At the single limit state function level, the reliability associated with g_1 < 0 in Failure Mode 1 is mainly sensitive to changes in c_2.

Fig. 7. Variation of system reliability index with the correlation coefficient.
The least significant random variable is very different from that for Failure Mode 1 is also sensitive to $G_w$. Similarly, the reliability associated with $g_6 \leq 0$ in Failure Mode 3 is quite sensitive to changes in $c_2$, $G_w$ and $\phi_2$. While it is insensitive to $\delta_1$ and $\theta_1$. Based on these results, it can be seen that the changes in the cohesion and friction angle have a significant influence on the computed reliability for all the three limit state functions. Accordingly, the determination of cohesion and friction angle with sufficient accuracy is of paramount importance for an adequate assessment of wedge stability. The sensitivity coefficient for $G_w$ also has a high positive value for all three limit state functions, which indicates that a good drainage system for the slope can increase the wedge stability. These findings correspond positively with common understanding of rock slope engineering practice and rock mechanics principles.

For the sensitivity analyses at the single failure mode level, $G_w$ and $\phi_1$, while $\phi_2$ is the least significant random variable. For $g_4 \leq 0$ in Failure Mode 2, $\phi_1$, $G_w$ and $\delta_1$ are significant random variables with high sensitivity coefficients, while $\delta_2$ is the least significant random variable. Similarly, the reliability associated with $g_6 \leq 0$ in Failure Mode 3 is quite sensitive to changes in $c_2$, $G_w$ and $\phi_2$. While it is insensitive to $\delta_1$ and $\theta_1$. Based on these results, it can be seen that the changes in the cohesion and friction angle have a significant influence on the computed reliability for all the three limit state functions. Accordingly, the determination of cohesion and friction angle with sufficient accuracy is of paramount importance for an adequate assessment of wedge stability. The sensitivity coefficient for $G_w$ also has a high positive value for all three limit state functions, which indicates that a good drainage system for the slope can increase the wedge stability. These findings correspond positively with common understanding of rock slope engineering practice and rock mechanics principles.

Further study shows the system reliability sensitivity coefficients for various wedge heights (Fig. 9). It can be observed from Fig. 9 that $G_w$ is the most significant variable with the highest sensitivity coefficient regardless of the changes in the wedge height. That is, the change in $G_w$ has a significant influence on the system reliability of the wedge. Therefore, designing a good drainage system for the slope is an effective way to improve the wedge stability. This conclusion is consistent with the conclusions drawn from the results at both the single limit state function level and the single failure mode level. In addition, the system reliability of the wedge is quite sensitive to $\theta_2$, especially for low heights of the wedge, which indicates that the wedge geometry is a key factor in the wedge stability analysis, emphasizing the importance of a good geological investigation of discontinuities in the rock mass. However, $\phi_2$ is the least significant factor in the single limit state function $g_6 \leq 0$ as shown in Table 4, and the sensitivity coefficient of $\delta_2$ in Failure Mode 1 as indicated in Fig. 8 is almost equal to zero. Note that the sensitivities of reliability results in basic random variables at different levels can differ considerably. This highly depends on the selected sensitivity analysis level. It can also be seen from Fig. 9 that $G_w$, $\delta_1$ and $\theta_2$ have larger sensitivity coefficients when the wedge height is below 20 m, while $c_2$ and $\delta_1$ appear to be the least significant factor. In addition, the reliability for Failure Mode 1 is also sensitive to $G_w$, $\phi_1$ and $c_1$. Compared with the sensitivity analysis results for $g_1 \leq 0$ at the single limit state function level, the significant random variables remain the same. However, the least significant random variable is very different from that for $g_1 \leq 0$.

![Fig. 8. Comparison between sensitivity coefficients of random variables for Failure Mode 1.](image1)

![Fig. 9. Comparison of system reliability sensitivity coefficients for random variables.](image2)

**Table 4**

Sensitivity coefficients and design points of basic random variables.

<table>
<thead>
<tr>
<th>Variable</th>
<th>$g_1 \leq 0$</th>
<th>$g_2 \leq 0$</th>
<th>$g_3 \leq 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$X'$</td>
<td>$X'$</td>
<td>$X'$</td>
</tr>
<tr>
<td>$c_1$ (kPa)</td>
<td>3.87E+01</td>
<td>-2.48E-01</td>
<td>4.82E+01</td>
</tr>
<tr>
<td>$c_2$ (kPa)</td>
<td>3.38E+01</td>
<td>-6.34E-01</td>
<td>4.20E+01</td>
</tr>
<tr>
<td>$\phi_1$</td>
<td>3.18E+01</td>
<td>-3.20E-01</td>
<td>4.58E+01</td>
</tr>
<tr>
<td>$\phi_2$</td>
<td>3.43E+01</td>
<td>-7.33E-02</td>
<td>3.50E+01</td>
</tr>
<tr>
<td>$\delta_1$</td>
<td>5.04E+01</td>
<td>1.31E+01</td>
<td>4.79E+01</td>
</tr>
<tr>
<td>$\delta_2$</td>
<td>4.84E+01</td>
<td>1.35E+01</td>
<td>4.81E+01</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>6.12E+01</td>
<td>-1.56E-01</td>
<td>4.28E+01</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>1.91E+01</td>
<td>-1.78E-01</td>
<td>2.17E+01</td>
</tr>
<tr>
<td>$G_w$</td>
<td>6.13E+01</td>
<td>5.81E-01</td>
<td>3.62E-01</td>
</tr>
</tbody>
</table>

*Note: $X'$ and $X$ represent the sensitivity coefficients and the design points, respectively. The symbol ‘*’ represents that the variable is not included in the limit state function.*
5. Summary and conclusions

This paper has proposed a methodology for evaluation of rock wedge stability, using a system reliability approach and considering multiple correlated failure modes. A probabilistic fault tree is used to model the system reliability of the wedge. The proposed N-dimensional equivalent method is employed to perform the reliability computation. An example is presented to illustrate the proposed methodology. Several conclusions can be drawn from this study:

(1) The stability of a wedge can be evaluated efficiently using the system reliability approach with an N-dimensional equivalent method. The system probabilities of failure for the wedge using the N-dimensional equivalent method are fairly consistent with those obtained from the Monte Carlo simulation method, which indicates that the proposed N-dimensional equivalent method is valid.

(2) The system reliability of a wedge with multiple correlated failure modes can be modelled using the probabilistic fault tree in a more rational way. The correlations between the different failure modes can be taken into consideration properly. Neglecting such correlations will result in an overestimated probability of failure of the wedge.

(3) The most likely failure mode can be identified using the proposed method, which could be important for improving design and reinforcement measures for rock slope engineering. Results show that the relative importance of different failure modes to the system reliability can differ considerably. In the example, Failure Modes 1 (biplane failure) and 2 (plane 1 failure only) are significantly more likely than Failure Modes 3 (plane 2 failure only) and 4 (floating failure). The ability of the proposed approach to quantify the relative importance of each failure mode is a valuable feature that can help the designer to establish priorities and decision making for rock slope engineering. It should be noted that the contribution of each failure mode to the system reliability is highly dependent on the wedge geometry, and the cohesion and friction angle of the discontinuities forming the wedge.

(4) The system reliability of the wedge stability increases with the negative correlation between the cohesion and friction angle. If such correlation is not taken into account, the system reliability of the wedge will be underestimated.

(5) The sensitivity analysis with respect to basic random variables can be conducted at three different levels, namely, single limit state function level, single failure mode level, and system reliability level. The sensitivity results are highly dependent on the selected sensitivity analysis level. In the example, at the system reliability sensitivity level, the water pressure parameter and the wedge geometry are significant variables with high sensitivity coefficients. Therefore, to improve the wedge stability effectively, a good drainage system for the particular slope should be designed and an adequate structural characterization of the rock mass should be conducted.

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