Impact of copula selection on geotechnical reliability under incomplete probability information

Xiao-Song Tang a, Dian-Qing Li a, b, Guan Rong a, Kok-Kwang Phoon b, Chuang-Bing Zhou a

a State Key Laboratory of Water Resources and Hydropower Engineering Science, Key Laboratory of Rock Mechanics in Hydraulic Structural Engineering, Ministry of Education, Wuhan University, 8 Donghu South Road, Wuhan 430072, PR China
b Department of Civil and Environmental Engineering, National University of Singapore, Blk E1A, #07-03, 1 Engineering Drive 2, Singapore 117576, Singapore

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Abstract

This paper aims to investigate the impact of copula selection on geotechnical reliability under incomplete probability information. The copula theory is introduced briefly. Thereafter, four copulas, namely Gaussian, Plackett, Frank, and No. 16 copulas, are selected to model the dependence structure between cohesion and friction angle. A copula-based approach is used to construct the joint probability density function of cohesion and friction angle with given marginal distributions and correlation coefficient. The reliability of an infinite slope and a retaining wall is presented to demonstrate the impact of copula selection on reliability. The results indicate that the probabilities of failure of geotechnical structures with given marginal distributions and correlation coefficient of shear strength parameters cannot be determined uniquely. The resulting probabilities of failure associated with different copulas can differ considerably. Such a difference increases with decreasing probability of failure. Significant difference in probabilities of failure could be observed for relatively small coefficients of variation of the shear strength parameters or a strong negative correlation between cohesion and friction angle. The Gaussian copula, often adopted out of expedience without proper validation, may not capture the dependence structure between cohesion and friction angle properly. Furthermore, the Gaussian copula may greatly underestimate the probability of failure for geotechnical structures.

Corresponding author. Tel.: +86 27 6877 2496; fax: +86 27 6877 2310.
E-mail address: dianqing@whu.edu.cn (D.-Q. Li).

For reliability analysis of geotechnical structures such slopes and retaining walls, parameters such as cohesion and friction angle are often treated as random variables. In principle, to evaluate the reliability of geotechnical structures exactly, the joint cumulative distribution function (CDF) or probability density function (PDF) of random variables must be known. In geotechnical engineering practice, however, the joint CDF or PDF is often unknown because of limited data from field test and laboratory test [23]. In most cases, only the marginal distributions and the covariance matrix are known [8,9,19,22]. Based on this incomplete probability information, the joint CDF or PDF of random variables cannot be determined uniquely when the actual joint probability distribution of random variables is a multivariate non-normal distribution. Therefore, characterization and simulation of multivariate distributions based on incomplete probability information remains an outstanding practical challenge [13,38,39].

Recently, copula theory [e.g., 32,35] has found widespread application for constructing joint probability distribution of multivariate data, particularly bivariate data. Copulas are functions that join multivariate distribution functions to their one-dimensional marginal distribution functions. There are many copulas in the literature such as Gaussian, t, Frank, Clayton, Gumbel and Plackett copulas. The difference among various copulas lies in the fact that each copula has its own dependence structure. Copulas provide a fairly general method for constructing multivariate distributions that satisfy some non-parametric measure of dependence and the prescribed marginal distributions. The copula theory has been extensively used for financial and hydrological applications [32,11,41]. In recent years, the copula theory has been applied to geotechnical engineering. For example, Bárdoesy and Li [5] explored the application of copulas as geostatistical methods for geostatistical interpolation, which are applied to a large scale groundwater quality measurement network in Baden-Württemberg. Kazianka and Pilk [17] proposed three different copula-based spatial interpolation methods and introduced geostatistical copula-based models to deal with random fields having discrete marginal distributions. Kazianka and Pilk [18] further proposed a Bayesian spatial copula model and a Bayesian copula-based spatial interpolation approach. Dithinde et al. [7] used the Gaussian copula to characterize the spatial dependence of rainfall data. Finally, the use of copula-based methods for engineering applications has been demonstrated in Kazianka and Pilk [19]. Kazianka et al. [20] demonstrated the use of copula-based models in rainfall-runoff modeling.
copula to capture the dependence structure between two curve-fitting parameters underlying the load–displacement curve of piles. Li et al. [21] used the copula approach to construct the joint PDF of two curve-fitting parameters underlying load–displacement curve of piles. Marchant et al. [30] introduced a more general multivariate function for the spatial analysis of soil properties based on copulas. Uzielli and Mayne [43,44] investigated the dependence among load–displacement model parameters underlying vertically loaded shallow footings on sands using copula.

In reliability analyses of slopes and retaining walls, the shear strength parameters, namely cohesion and friction angle, are usually treated as random variables [4,26,27,28]. Furthermore, it is widely accepted that cohesion and friction angle are negatively correlated [6,4,15,27,25,20]. For reliability evaluations of a slope and a retaining wall involving correlated non-normal variables, the Nataf transformation [34] is often employed to transform the correlated non-normal variables to independent standard normal variables. Essentially, the Nataf transformation adopts a Gaussian copula for modeling dependence structure between variables. In other words, the reliability analyses of a slope and a retaining wall involving correlated cohesion and friction angle are generally based on the implicit assumption that the Gaussian copula is adequate to describe the dependence structure between cohesion and friction angle, and the resultant joint distribution of cohesion and friction angle is a multivariate normal distribution. However, the Gaussian copula may not provide the best fit to the dependence structure between cohesion and friction angle in all soils. A statistical study is conducted in this paper to demonstrate that other copulas may be more appropriate. Hence, it is of practical interest to study and delineate the circumstances under which the copula structure would affect the probabilities of failure for slope and retaining wall significantly.

This paper aims to investigate the impact of copula selection on reliability for a slope and a retaining wall under incomplete probability information where the marginal distributions and correlation coefficient of shear strength parameters are known. To achieve such a goal, this article is organized as follows. In Section 2, copula theory is introduced briefly. Thereafter, the capability of the Gaussian copula to model the dependence structure between cohesion and friction angle is studied based on four datasets of measured shear strength parameters from field tests. In Section 3, nominal factors of safety for a slope and a retaining wall are defined, and reliability analyses of an infinite slope and a semi-gravity retaining wall are carried out. Furthermore, the effects of copulas on the probabilities of failure of the infinite slope and the semi-gravity retaining wall are presented.

2. Modeling of joint CDF of cohesion and friction angle using copulas

As mentioned previously, copulas are functions that couple a multivariate distribution to its one-dimensional marginal distributions. Alternatively, copulas are multivariate distribution functions whose one-dimensional marginal distributions are uniform on the interval of [0, 1]. Since Sklar’s theorem is central to the copula theory and is the foundation of many applications of that theory, such a theorem is first introduced herein.

2.1. Copulas

Sklar’s theorem (e.g., [35]). Let \( F(x_1, x_2, \ldots, x_n) \) be a joint CDF with marginal CDFs \( F_1(x_1), F_2(x_2), \ldots, F_n(x_n) \). Then there exists an \( n \)-dimensional copula \( C \) such that for all real \( x_1, x_2, \ldots, x_n \),

\[
F(x_1, x_2, \ldots, x_n) = C[F_1(x_1), F_2(x_2), \ldots, F_n(x_n)]
\]

If \( C \) is a copula and \( F_1(x_1), F_2(x_2), \ldots, F_n(x_n) \) are CDFs, then the function \( F(x_1, x_2, \ldots, x_n) \) defined by Eq. (1) is a joint distribution function with marginals \( F_1(x_1), F_2(x_2), \ldots, F_n(x_n) \). Sklar’s theorem essentially states that the joint probability distribution of random variables can be expressed in terms of a copula function and their marginal distributions. In other words, fitting a joint probability distribution to measured data involves two steps: (1) determining the statistical parameters of marginal distributions from measured data, and (2) identifying the copula which provides the best fit to the measured dependence structure from a set of candidate copulas in some sense (e.g., AIC or BIC).

According to Sklar’s theorem, the bivariate joint CDF of two random variables \( X_1 \) and \( X_2 \) can be given by

\[
F(x_1, x_2) = C(F_1(x_1), F_2(x_2); \theta) = C(u_1, u_2; \theta)
\]

in which \( F(x_1, x_2) \) is the joint CDF of \( X_1 \) and \( X_2 \); \( u_1 = F_1(x_1) \) and \( u_2 = F_2(x_2) \) are the corresponding marginal distributions of \( X_1 \) and \( X_2 \), respectively; \( C(u_1, u_2; \theta) \) is the copula function in which \( \theta \) is the copula parameter describing the dependence between \( X_1 \) and \( X_2 \). By taking derivatives of Eq. (2), the bivariate PDF \( f(x_1, x_2) \) of \( X_1 \) and \( X_2 \) can be obtained as (e.g., [32])

\[
f(x_1, x_2) = f_1(x_1)f_2(x_2)c(F_1(x_1), F_2(x_2); \theta)
\]

where \( c(F_1(x_1), F_2(x_2); \theta) \) is the copula density function, which is given by

\[
c(F_1(x_1), F_2(x_2); \theta) = \frac{\partial^2 C(u_1, u_2; \theta)}{\partial u_1 \partial u_2}
\]

Theoretically, the joint CDF and PDF of \( X_1 \) and \( X_2 \) can be determined by Eqs. (2) and (3) if the marginal distributions of \( X_1 \) and \( X_2 \) and the copula function are known.

The copula parameter \( \theta \) can be determined through the Pearson linear correlation coefficient or rank correlation coefficient such as Spearman and Kendall correlation coefficients. The Pearson correlation coefficient is commonly used as a measure of dependence between variables in geotechnical engineering. The reason is that the popularity of the Pearson correlation coefficient is strongly related to the prevalence of multivariate normal distributions in modeling multivariate random data as mentioned previously. However, the Pearson correlation coefficient is invariant under strictly monotonic linear transformations only, and thus the dependence measure needs to be evaluated for each nonlinear transformation. Furthermore, if random variables are not jointly Gaussian or some nonlinear transformations are used, the advantage of the Pearson correlation coefficient is lost and its use would be misleading. Unlike the Pearson correlation coefficient, the rank correlation coefficient is rank-dependent and invariant with respect to strictly monotonic transformations, which allows for a unique dependence measure for all such transformed variables. Moreover, the rank correlation coefficient can be expressed in terms of a copula function and it serves as an intrinsic dependence measure in copula modeling. Nonetheless, to be in correspondence with the geotechnical engineering practice, the Pearson correlation coefficient is adopted to determine \( \theta \) by keeping the aforementioned strengths and weaknesses associated with the two correlation coefficients in mind. According to the definition of Pearson correlation coefficient (e.g., [2]), the following integral relation between \( \rho \) and \( \theta \) can be obtained,

\[
\rho = \frac{\text{Cov}(X_1, X_2)}{\sigma_1 \sigma_2} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \frac{1}{\sigma_1} - \frac{\mu_1}{\sigma_1} \right] \left[ \frac{1}{\sigma_2} - \frac{\mu_2}{\sigma_2} \right] f_1(x_1)f_2(x_2)c(F_1(x_1), F_2(x_2); \theta) dx_1 dx_2
\]

where \( \mu_1 \) and \( \mu_2 \) are the means of \( X_1 \) and \( X_2 \), respectively; \( \sigma_1 \) and \( \sigma_2 \) are the standard deviations of \( X_1 \) and \( X_2 \), respectively; \( \text{Cov}(X_1, X_2) \) is the covariance between \( X_1 \) and \( X_2 \). For given marginal distributions of \( X_1 \) and \( X_2 \), and correlation coefficient \( \rho \) between \( X_1 \) and \( X_2 \), the
The Mohr–Coulomb failure criterion is commonly adopted to model the failure envelope of soils and rocks. The intercept of the failure envelope with the shear strength axis is the cohesion, and the slope of the failure envelope is the internal friction angle. Hence, the Mohr–Coulomb criterion is described by two parameters: cohesion \( c \) and friction angle \( \phi \). It is widely accepted that there exists a negative correlation between cohesion and friction angle, because of linearization of the nonlinear Mohr–Coulomb failure envelope. Table 1 summarizes the correlation coefficient between cohesion and friction angle reported in the literature. In practice, the Gaussian copula is often used to model the dependence structure between cohesion and friction angle \[38,20\]. It should be noted that the Gaussian copula is commonly adopted out of practical expediency without rigorous validation. To examine the adequacy of this expedient assumption, measured data of cohesion and friction angle obtained from field tests in four hydropower stations in China are used as shown in Table 2 \[45\]. The strength data belong to: (1) a weak intercalated layer at Jiangkou Hydropower Station (referred to as WI-JK hereafter), (2) a weak intercalated layer at Ankang Hydropower Station (WI-AK), (3) a concrete and bedrock cemented surface at Shuikou Hydropower Station (CS-SK), and (4) a concrete and bedrock cemented surface at Ertan Hydropower Station (CS-ET). The sample sizes \( N \) are 16, 25, 25, and 42, respectively.

In order to visualize the dependence structure of the measured data \( (c, \phi) \), the measured data in original space should be transformed into the standard uniform random vector \( U = (U_1, U_2) \). For such a purpose, the empirical distributions of \( c \) and \( \phi \) are adopted in this study. \( U_1 \) and \( U_2 \) are defined as (e.g., \[35\])

\[
\begin{align*}
U_{1i} &= \frac{\text{rank}(c_i)}{N + 1}, \\
U_{2i} &= \frac{\text{rank}(\phi_i)}{N + 1},
\end{align*}
\]

in which \( \text{rank}(c_i) \) [or \( \text{rank}(\phi_i) \)] denotes the rank of \( c_i \) [or \( \phi_i \)] among the list \( \{c_1, ..., c_N\} \) [or \( \{\phi_1, ..., \phi_N\} \)] in an ascending order. The scatter plots for \( U_1 \) versus \( U_2 \) are shown in Fig. 1. It can be seen that there is a strongly negative dependence between \( U_1 \) and \( U_2 \). Furthermore, the samples of \( U_1 \) and \( U_2 \) for the four datasets are basically symmetrical with respect to the diagonal line of a unit square, particularly evident in the CS-ET dataset with the largest sample size.

For illustrative purposes, the Gaussian copula, Plackett copula, Frank copula, and No. 16 copula \[35\] are examined in this study. The above four copulas, along with the range of the \( \theta \) parameter are listed in Table 3. Since the Frank copula and No. 16 copula belong to the Archimedian copula class, their generators are also provided in Table 3. Note that the aforementioned copulas except the No. 16 copula are symmetric copulas. However, the No. 16 copula is approximately symmetric when the negative correlation is strong. The selected four copulas can describe both positive and negative dependence and the absolute value of the correlation coefficient can approach 1. These features are useful for describing the dependence structure between \( c \) and \( \phi \) shown in Fig. 1. It should be pointed out that since a copula function only describes the dependence structure between variables and is independent of the underlying marginal distributions, all the above four copulas can incorporate different marginal distributions in a multivariate distribution.

Having selected the candidate copulas, the next step is to determine the copula parameters. As mentioned in Section 2.1, the Pearson correlation coefficient is used to determine the copula parameters. The Pearson correlation coefficient between \( c \) and \( \phi \), denoted as \( \rho_{c, \phi} \), is defined as (e.g., \[31\])

\[
\rho_{c, \phi} = \frac{\sum_{i=1}^{N}(c_i - \mu_c)(\phi_i - \mu_\phi)}{\sqrt{\sum_{i=1}^{N}(c_i - \mu_c)^2 \sum_{i=1}^{N}(\phi_i - \mu_\phi)^2}}
\]

in which \( (\phi_i, \phi_i) \) denotes a pair of \( c \) and \( \phi \) values; \( N \) is the sample size; \( \mu_c \) and \( \mu_\phi \) denote the sample means of \( c \) and \( \phi \), respectively. Based on the datasets and measured \( (c, \phi) \) given in Table 2, the Pearson correlation coefficients between \( c \) and \( \phi \) can be obtained using Eq. (7), which are shown in the second column of Table 4. By substituting \( \rho_{c, \phi} \) into Eq. (5), the copula parameters \( \theta \) associated with the four copulas can be determined, which are also listed in Table 4.

After obtaining the copula parameters, the copula functions and copula density functions shown in Table 3 can be determined. Thereafter, the best-fit copula is identified based on the minimum value of the Akaike Information Criterion (AIC) \[1\] and the Bayesian Information Criterion (BIC) \[42\], which are respectively defined as

\[
\text{AIC} = -2\sum_{i=1}^{N} \ln c(u_{1i}, u_{2i}) + 2k
\]

and

\[
\text{BIC} = -2\sum_{i=1}^{N} \ln c(u_{1i}, u_{2i}) + k\ln N
\]

where \( k \) is the number of copula parameters; \( A \) copula producing the smallest AIC value or BIC value is considered to be the best-fit copula. For the four selected copulas, all of them are single parameter copulas. Therefore, \( k = 1 \) is used in Eqs. (8) and (9). For the datasets considered, substituting \( (u_{1i}, u_{2i}) \) obtained from Eq. (6) into Eqs. (8) and (9), the AIC and BIC values for the four copulas selected can be obtained, which are also shown in Table 4. Note that both the AIC and BIC values consistently indicate that the No. 16 copula is the best-fit copula for the WI-JK dataset. For the WI-AK and CS-SK datasets, the Plackett copula is the best-fit copula. The Frank copula is the best-fit copula for the CS-ET dataset. All of them are better fits to measured data than the Gaussian copula based on AIC and BIC. The above results indicate that the Gaussian copula may not provide the best fit to the dependence between \( c \) and \( \phi \). It is natural to question if this best fit copula or less than best fit Gaussian copula will produce significantly different probabilities of failure when applied to reliability analyses. Two examples are studied in the next section with this in mind.
It is well known that the shear strength parameters $c$ and $\phi$ have a significant influence on slope reliability. Therefore, both $c$ and $\phi$ are taken as random variables. Following Jimenez-Rodriguez et al. [16], a lognormal distribution is adopted to model the distributions of $c$ and $\phi$. The other three parameters, namely $H$, $\gamma$, and $\alpha$, are assumed as deterministic so that the negative correlation between $c$ and $\phi$ can be studied without interference from other random variables. The parameters recommended by Griffiths et al. [12] are adopted as summarized below. The mean and coefficient of variation (COV) of $c$ are 12 kPa and 0.4, respectively. The mean and COV of $\phi$ are $30^\circ$ and 0.2, respectively. The deterministic parameters are $\gamma = 17$ kN/m$^3$, $H = 5$ m, and $\alpha = 30^\circ$. Based on the correlation coefficients $\rho_{c,\phi}$ reported in the literature, as shown in Table 1, a correlation coefficient of $\rho_{c,\phi} = -0.5$ is used to account for the effect of correlation on slope reliability.

3.2. Retaining wall example

A semi-gravity retaining wall [26] shown in Fig. 3 is employed to investigate the reliability of retaining wall under incomplete probability information. Generally, three geotechnical failure modes need to be considered in the design of a semi-gravity retaining wall: (1) overturning of the wall about its toe, (2) sliding along

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Note: WI-JK = Weak interlayer in rock mass of Jiangkou Hydropower Station; WI-AK = weak interlayer in rock mass of Ankang Hydropower Station; CS-SK = cemented surface between concrete and bedrock of Shuikou Hydropower Station; CS-ET = cemented surface between concrete and bedrock of Ertan Hydropower Station.
its base, and (3) bearing capacity failure of the foundation soil. The overturning failure mode is examined below, as the effect of the strength parameters on the factor of safety for this mode is strongly nonlinear. The existing deterministic approach evaluates a lumped factor of safety against overturning failure about the wall’s toe as [26]
where $M_{resisting}$ and $M_{overturning}$ denote the actual resisting moments and overturning moments, respectively; $W_1$ and $W_2$ are the component weights of the retaining wall, with horizontal lever distances $Arm_1$ and $Arm_2$ respectively, measured from the toe of the wall; $P_a$ is the active earth thrust with a vertical lever distance $Arma$. In this paper, Rankine’s theory is used to compute $P_a$, which is based on the assumption that the back of the retaining wall is frictionless. For backfill with cohesion $c$ and internal friction angle $\phi$, $P_a$ is given by

$$P_a = \frac{1}{2} \gamma_{wall} b H - 2 c H \sqrt{K_a} + \frac{2 c^2}{\gamma_{wall}}$$

where $K_a$ is the coefficient of active earth pressure; $\gamma_{wall}$ is the unit weight of the retaining wall concrete. It is known that the shear strength parameters $c$ and $\phi$ of the retained soil have a significant influence on the probability of overturning failure [3]. Therefore, both $c$ and $\phi$ are treated as random variables. Following Jimenez-Rodriguez et al. [16], a lognormal distribution is adopted to model the distributions of $c$ and $\phi$ again. The other five parameters, namely $H$, $a$, $b$, $\gamma_{wall}$, and $\gamma_{wall}$, are assumed as constants so that the negative correlation between $c$ and $\phi$ can be studied without interference from other random variables. Using the coefficients of variation (COVs) for $c$ and $\phi$ in Table 5 as a reference, the mean and COV of $c$ are assumed to be 12 kPa and 0.4, respectively. The mean and COV of $\phi$ are assumed to be 20° and 0.2, respectively. The deterministic parameters are $H = 6$ m, $a = 0.4$ m, $b = 1.4$ m, $\gamma_{wall} = 18$ kN/m$^3$, and $\gamma_{wall} = 24$ kN/m$^3$. Based on the correlation coefficients $q_{c,\phi}$ reported in the literature, as shown in Table 1, a correlation coefficient of $q_{c,\phi} = 0.5$ is selected to account for the effect of correlation on the probability of overturning failure.

3.3. Probability of failure using direct integration

The same performance function is adopted for the infinite slope and retaining wall examples:

$$g(c, \phi) = FS(c, \phi) - 1$$

in which $FS(c, \phi)$ is determined by Eq. (10) for the infinite slope example and Eq. (11) for the retaining wall example. It should be noted that the performance function of the retaining wall example
is a cubic equation with respect to cohesion $c$. On the other hand, the performance function of the infinite slope example is linear with respect to $c$.

In the current reliability literature, many reliability methods such as the first-order reliability method (FORM), Second-order reliability method (SORM), and Monte Carlo simulation (MCS), are available for determining the probability of failure, i.e., probability of $g(c, \phi)$ less than zero. To remove the errors such as error resulting from linearization at design point underlying the FORM and SORM, and statistical error due to sampling numbers underlying the MCS, a direct integration method is adopted to determine the probability of failure. In this way, the effect of different copulas on the probability of failure can be identified accurately. The formulae for calculating the probabilities of failure of the infinite slope and the retaining wall using direct integration are presented in the Appendix.

3.4. Nominal factors of safety for infinite slope and retaining wall

It is clear that a nominal factor of safety computed by substituting mean values for the random variables in Eqs. (10) and (11) cannot account for the coefficients of variation of shear strength parameters and the correlation between cohesion and friction angle. To account for these statistics approximately, nominal factors of safety involving cautious estimates of the strength parameters are introduced below for the infinite slope and the retaining wall. Following the Eurocode 7 practice [36], a 5% fractile value of the factors of safety as shown in Eqs. (10) and (11) are defined as the nominal factors of safety for the infinite slope and retaining wall, respectively. In this paper, the nominal factor of safety, $FS_n$, is obtained from simulations as illustrated below using the Plackett copula, which is the most frequently identified to be the best-fit copula as shown in Section 2.2.

The algorithm for simulating the nominal factor of safety consists of the following five steps:

1. Simulate two independent standard uniform random variables $V_{m,2} = [V_1, V_2]$. This can be obtained from MATLAB using: $V_{m,2} = \text{rand}(m, 2)$ and MATLAB function $\text{rand('state', 1)}$ is used to fix the initial seed. In this study, a sample size of 10^6 is adopted for the simulation.
2. Solve $\theta$ from Eq. (5) using the given $p_{c,0}$ and marginal distributions underlying the shear strength parameters. Then set $\sigma_0 = \sigma_2(1 - \sigma_2), \quad b_0 = \sigma_2(1 - \sigma_2), \quad c = 2a_0(\sigma_2 + 1 - \sigma_2) + \theta(1 - 2a_0), \quad \text{and} \quad d = \sqrt{b_0(\theta + 4a_0)(1 - \sigma_2)(1 - \sigma_2)}$.
3. Set $u_1 = \sigma_1$ and $u_2 = [c - (1 - 2\sigma_2)]d/2b_0$. Then, the correlated standard uniform vector $U_{m,2}$ is obtained as $U_{m,2} = [U_1, U_2]$ belonging to the Plackett copula.
4. Let $X_{m,2} = (c, \phi) = (F_1^{-1} (U_1), F_2^{-1} (U_2))$ in which $F_1^{-1} (\cdot)$ and $F_2^{-1} (\cdot)$ are the inverse CDFs of $c$ and $\phi$, respectively.
5. Substitute $X_{m,2} = (c, \phi)$ into Eq. (10) or Eq. (11). The factor of safety $FS_{n,1}$ for the infinite slope and the retaining wall is obtained. The 5% fractile value of the simulated factors of safety is obtained using the MATLAB function $\text{quantile}(\text{FS}, 0.05)$, which is taken as $FS_n$.

It is evident from the above simulation procedures that $FS_n$ is a function of deterministic parameters (e.g., depth of soil above bedrock, slope inclination) and statistical parameters (e.g., coefficient of variation, correlation coefficient). At first glance, there are similarities between $FS_n$ and $p_f$. Both quantities are derived from the cumulative distribution function of the factor of safety. The former is a fixed percentile value while the latter is the percentile corresponding to $FS < 1$. In the parametric studies presented below, the variation of $p_f$ with various deterministic and statistical parameters is studied by plotting against $FS_n$ rather than the mean factor of safety. There are two reasons for this more complicated choice. First, a single horizontal axis based on $FS_n$ can be applied in all parameter studies, thus providing a unified and concise presentation of the results. Second, $FS_n$ is closer to the nominal factor of safety computed in practice, because engineers will use lower bound strength values, rather than mean values. The acceptable nominal factors of safety for slopes fall between 1.2 and 1.5. It is possible to relate $p_f$ to these acceptable limits in an approximate way when $p_f$ is plotted against $FS_n$. In this way, the probabilistic results can be referenced to the more familiar nominal factors of safety and be made more meaningful to the engineers.

3.5. Comparison of reliability results produced by different copulas

The effect of copulas on the probability of failure is studied systematically based on three factors: (1) geometrical parameters ($H, x$) for the infinite slope (Fig. 2) and ($a, b$) for the retaining wall (Fig. 3), (2) COV scaling factor, $\lambda_{c,b}$ defined as: $\text{COV}_c = 0.4/\lambda_{c,b}$ and $\text{COV}_b = 0.2/\lambda_{c,b}$, and (3) $\rho_{c,b}$. In the parametric studies shown in Figs. 4 and 5, each factor is varied over a range of values shown in the figure caption while the other parameters remain unchanged. To facilitate comparisons between the four subplots in Figs. 4 and 5, changes in each factor are presented in a uniform way as changes in the nominal factor of safety as mentioned in Section 3.4. In other words, the horizontal axes are identical, although changes in the nominal factor of safety are caused by different factors in each subplot.

3.5.1. Effect of geometrical parameters on the probability of slope failure

This sub-section studies the effect of geometrical parameters ($H, x$) on infinite slope reliability. The results for the parametric study of these two geometrical parameters are presented below separately.

Table 5

<table>
<thead>
<tr>
<th>Variables</th>
<th>COV (%)</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c$</td>
<td>10–80</td>
<td>Harr [14]</td>
</tr>
<tr>
<td></td>
<td>40</td>
<td>Bacher and Christian [4]</td>
</tr>
<tr>
<td></td>
<td>10–55</td>
<td>Phoon [40]</td>
</tr>
<tr>
<td>$\phi$</td>
<td>5/20</td>
<td>Harr [14]</td>
</tr>
<tr>
<td></td>
<td>5–15</td>
<td>Phoon [40]</td>
</tr>
<tr>
<td></td>
<td>5–15</td>
<td>Mollon et al. [33]</td>
</tr>
<tr>
<td></td>
<td>20</td>
<td>Penalba et al. [37]</td>
</tr>
</tbody>
</table>

Table 5: Reported coefficients of variation of cohesion and friction angle of soils and rocks.
(1) Effect of depth of soil above bedrock (H) on the probability of slope failure

Applying the direct integration method as shown in the Appendix, the probability of failure of the infinite slope can be obtained. The nominal factor of safety is calculated by the steps presented in Section 3.4. Fig. 4a shows the probabilities of failure on log scale produced by different copulas for various values of $F_{Sn}$ as a result of varying the depth of soil above bedrock, $H$. Note that the probabilities of failure can differ considerably. The Gaussian copula results in the smallest probability of failure among the selected four copulas. On the other hand, the No. 16 copula leads to the largest probability of failure. Therefore, the Gaussian copula, often used for modeling the joint probability distribution of correlated shear strength parameters, will significantly overestimate the slope reliability, which is unconservative for slope safety assessment. As $H$ decreases, $F_{Sn}$ increases, while the probability of failure decreases.

To investigate the difference in probabilities of failure produced by different copulas quantitatively, Table 6 presents the relative differences in probabilities of failure produced by different copulas and nominal factors of safety. In Table 6, the probability of failure associated with the Gaussian copula is taken as a reference case. Note that, the values in Table 6 are calculated by $p_{f}/p_{f,\text{Gaussian}}$ in which $p_f$ is the probability of failure produced by the Plackett copula, Frank copula, or No. 16 copula, and $p_{f,\text{Gaussian}}$ is the probability of failure produced by the Gaussian copula. It is evident that the differences in probabilities of failure produced by different copulas are significant, especially for $F_{Sn} = 1.17$. The ratios $p_{f}/p_{f,\text{Gaussian}}$ associated with $F_{Sn} = 1.17$ are 2.87, 2.21, and 12.16 for the Plackett, Frank, and No. 16 copulas, respectively. These results further indicate that the probabilities of failure of the infinite slope can differ fairly significantly. Thus, it is important to identify the best-fit copula underlying the measured data in hand. Otherwise, the misuse of copula may cause unacceptable errors in probability of failure. In addition, the differences in probabilities of failure produced by a specified copula increase with increasing $F_{Sn}$ or decreasing probability of failure, especially for the No. 16 copula. For example, the ratio $p_{f}/p_{f,\text{Gaussian}}$ associated with the No. 16 copula increases from 1.85 to 12.16 when $F_{Sn}$ increases from 1.03 to 1.17. Note that $F_{Sn} = 1.17$ is probably closer to the design target for engineered slopes in practice, while $F_{Sn} = 1.03$ may appear in natural slopes. Hence, there is practical significance to the above numerical observations.

(2) Effect of slope inclination ($\alpha$) on the probability of slope failure

Fig. 4b shows the probabilities of failure on log scale produced by different copulas for various values of $F_{Sn}$ as a result of varying the slope inclination, $\alpha$. Similar to the results shown in Fig. 4a, the probabilities of failure produced by different copulas are different. The Gaussian copula also produces the smallest probability of failure among the selected four copulas, which
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is unconservative for slope safety assessment. When $\alpha$ decreases from 30.0° to 26.5°, the $F_{Sn}$ increases from 1.03 to 1.17, and the corresponding probability of failure produced by the Gaussian copula decreases from 3.03E-02 to 2.52E-03. The relative differences in probabilities of failure produced by different copulas and nominal factors of safety are listed in the fifth to seventh columns of Table 6. The same conclusions as those drawn from Fig. 4a can be made. The ratios $p_{f}/p_{fGaussian}$ associated with $F_{Sn} = 1.17$ are 2.63, 2.23, and 10.87 for the Plackett, Frank, and No. 16 copulas, respectively. These results imply that the differences in probabilities of failure produced by different copulas are significant. For a specified copula, the differences in probabilities of failure increase as $F_{Sn}$ increases or the probability of failure decreases.

3.5.2. Effect of geometrical parameters on the probability of retaining wall overturning failure

The results for the parametric study of geometrical parameters ($a$, $b$) on retaining wall reliability are presented here. Similarly, applying the direct integration method as shown in the Appendix, the probability of failure of the retaining wall can be obtained. Fig. 5a and b shows the probabilities of failure on log scale produced by different copulas for various values of $F_{Sn}$ as a result of varying $a$ (minimum wall width at the top) and $b$ (maximum wall width at the base). The same conclusions as those drawn from Fig. 4a and b can be made although the strength parameters ($c$, $\phi$) appear in a highly nonlinear performance function. In addition, the nominal factors of safety increase with increasing values of $a$ or $b$ as to be expected.
3.5.3. Effect of COV of shear strength parameters on the probability of failure

In this sub-section, the effect of COV of shear strength parameters on infinite slope and retaining wall reliability is investigated. The results for the parametric study of COV of shear strength parameters are presented separately for the infinite slope and the retaining wall.

(1) Infinite slope example

Fig. 4c shows the probabilities of failure on log scale produced by different copulas for various values of $F_{S_n}$ as a result of varying the COV scaling factor, $k_{c,\phi}$. The results shown in Fig. 4a and b, the probabilities of failure produced by different copulas differ considerably. Again, the Gaussian copula produces the smallest probability of failure among the selected four copulas. As the COV$_c$ and COV$_\phi$ decrease (or $k_{c,\phi}$ increases), $F_{S_n}$ increases and the corresponding probability of failure decreases. However, the probability of failure is more sensitive to $F_{S_n}$ than those shown in Fig. 4a and b. For instance, when the $F_{S_n}$ ranges from 1.03 to 1.17, the probabilities of failure for the Gaussian copula are within the range of $[3.03E-02, 2.52E-03]$ as shown in Fig. 4a and b, which is significantly wider than $[3.03E-02, 2.48E-03]$ as shown in Fig. 4a or $[3.03E-02, 2.52E-03]$ as shown in Fig. 4b. The relative differences in probabilities of failure produced by different copulas and nominal factors of safety are listed in the eighth to tenth columns of Table 6. When the same $F_{S_n}$ is adopted, the differences in probabilities of failure associated with different copulas are significant. For the Plackett, Frank, and No. 16 copulas, the ratios $p_{f}/p_{f_{Gaussian}}$ associated with $F_{S_n} = 1.17$ are 12.08, 8.94, and 246, respectively. The probability of failure for the Gaussian copula is 245 times smaller than that for the No. 16 copula. Additionally, for a specified copula, the differences in probabilities of failure increase as the $F_{S_n}$ increases.

(2) Retaining wall example

Similarly, the probabilities of failure on log scale produced by different copulas are shown in Fig. 5c for the retaining wall example. It can also be seen that $F_{S_n}$ increases as $k_{c,\phi}$ increases. The resulting probability of failure decreases significantly. Furthermore, the probability of failure is more sensitive to $F_{S_n}$ compared with Fig. 5a and b.

3.5.4. Effect of correlation between cohesion and friction angle on the probability of failure

This sub-section deals with the effect of correlation between cohesion and friction angle on infinite slope and retaining wall reliability. The results for the parametric study of correlation coefficient between cohesion and friction angle on infinite slope and retaining wall reliability are presented below.

(1) Infinite slope example

Fig. 4d shows the probabilities of failure on log scale produced by different copulas for various values of $F_{S_n}$ as a result of varying $\phi_{\text{cor}}$. The results are qualitatively the same as those shown in Fig. 4a–c, but the probabilities of failure associated

![Fig. 6](image-url) Scatter plots of $c$ and $\phi$ generated by different copulas for infinite slope stability with $\phi_{\text{cor}} = -0.5$, COV$_c = 0.4$ and COV$_\phi = 0.2$. 

Table 6. When the same $F_{S_n}$ is adopted, the differences in probabilities of failure associated with different copulas are significant. For the Plackett, Frank, and No. 16 copulas, the ratios $p_{f}/p_{f_{Gaussian}}$ associated with $F_{S_n} = 1.17$ are 12.08, 8.94, and 246, respectively. The probability of failure for the Gaussian copula is 245 times smaller than that for the No. 16 copula. Additionally, for a specified copula, the differences in probabilities of failure increase as the $F_{S_n}$ increases.

(2) Retaining wall example

Similarly, the probabilities of failure on log scale produced by different copulas are shown in Fig. 5c for the retaining wall example. It can also be seen that $F_{S_n}$ increases as $k_{c,\phi}$ increases. The resulting probability of failure decreases significantly. Furthermore, the probability of failure is more sensitive to $F_{S_n}$ compared with Fig. 5a and b.
with the Gaussian copula can be several orders of magnitude smaller than those associated with the other three copulas, especially for a strong negative correlation between cohesion and friction angle. Furthermore, the probabilities of failure associated with the Gaussian copula are very sensitive to the change of $F_{SN}$. When $F_{SN}$ varies from 1.03 to 1.17, the probability of failure associated with the Gaussian copula decreases from $3.03 \times 10^{-3}$ to $1.64 \times 10^{-7}$ (more than 4 orders of magnitude!). The relative differences in probabilities of failure associated different copulas and nominal factors of safety are listed in the last three columns of Table 6. The same conclusions as those drawn from the other three cases can also be made. However, the ratios $p_f/p_{f_{Gaussian}}$ associated with $F_{SN} = 1.17$ are significantly larger than those for the other three cases. For instance, the ratios $p_f/p_{f_{Gaussian}}$ associated with $F_{SN} = 1.17$ are 1.43E4, 6.84E3, and 4.61E4 for the Plackett, Frank, and No. 16 copulas, respectively.

3.6. Discussions

Based on the above results, it can be concluded that the probabilities of slope failure and the probabilities of overturning failure for retaining wall associated with the four selected copulas differ greatly, especially when different COVs of shear strength parameters or correlation coefficients between cohesion and friction angle are used. In this section, we will present some discussions to explain such differences in probability of failure through the following two ways. First, a comparison among simulated samples of cohesion and friction angle associated with various copulas is carried out. Second, relative locations between the limit state surfaces and the joint PDF isolines of cohesion and friction angle are investigated. Since the results for the infinite slope and the retaining wall are quite similar, only the results for the infinite slope are presented below.

Generally, it is much easier to appreciate a multivariate distribution from its simulated realizations. For illustration, only the simulated samples of cohesion and friction angle from the selected four copulas for $\mu_{c,a} = -0.5$, $COV_c = 0.4$ and $COV_\phi = 0.2$ are shown in Fig. 6, in which the sample size is 1000. The contour lines of constants $F_S = 1.0$, 1.2 and 1.5 for a representative slope with $H = 5$ m, $\alpha = 30^\circ$ and $\gamma = 17 \text{kN/m}^3$ are also plotted in Fig. 6. Again, the numbers ($N$) of samples falling in the regions associated with $F_S < 1.0$, $1.0 < F_S < 1.2$, $1.2 < F_S < 1.5$ and $F_S > 1.5$ are shown in...
the region associated with the other three copulas. Fig. 7 shows the joint PDF isolines of shear copula results in the smallest probability of failure. The Gaussian copula leads to the largest probability of failure while the Gaussian copulas, respectively. It is evident from these results that the No. 16 copula results in the smallest probability of failure while the Gaussian copula leads to the largest probability of failure.

To make a better comparison between the Gaussian copula and the other three copulas, Fig. 7 shows the joint PDF isolines of shear strength parameters associated with the four copulas selected. The joint PDF isoline associated with the Gaussian copula is plotted using a dashed line. For illustration, a representative PDF isoline value of 0.001 is used, which envelopes nearly the whole domain where simulated samples may appear. Note that the shape of such an isoline is similar to that of the scatter plots of simulated samples. It is evident that the joint PDFs of the shear strength parameters associated with different copulas differ considerably, especially between the Gaussian and No. 16 copulas. Such a difference further leads to the difference in probability of failure of the infinite slope between Gaussian copula and the other three copulas, as shown in Fig. 4.

To investigate the effect of relative locations between the limit state surfaces and the joint PDF isolines of cohesion and friction angle on slope reliability, three limit state surfaces are considered. They are associated with three slopes defined by: (1) \( H = 5.0 \text{ m}, \alpha = 30.0^\circ \) and \( F_{Sn} = 1.03 \), (2) \( H = 3.3 \text{ m}, \alpha = 30.0^\circ \) and \( F_{Sn} = 1.17 \), and (3) \( H = 5.0 \text{ m}, \alpha = 26.5^\circ \) and \( F_{Sn} = 1.17 \). The respective limit state surfaces are referred to as “Limit state I”, “Limit state II” and “Limit state III” in Fig. 7. It can be seen that the shift of \( F_{Sn} \) from 1.03 to 1.17 is due to the shift of \( H \) from 5.0 m to 3.3 m or \( \alpha \) from 30.0° to 26.5°. Similarly, the shift of \( F_{Sn} \) from 1.03 to 1.17 is caused by the shift of limit state surfaces from I to II or from I to III. Consequently, the probabilities of failure of the infinite slope decrease when \( F_{Sn} \) increases from 1.03 to 1.17 although the joint PDF of the shear strength parameters remains unchanged.

Fig. 8 shows the joint PDF isolines of the shear strength parameters associated with different copulas for different copulas for infinite slope stability with \( \mu_c = 0.5, \text{COV}_c = 0.2 \) and \( \text{COV}_\phi = 0.1 \). Note that the same PDF isoline value of 0.001 is adopted. Compared with Fig. 7, the boundary of the joint PDF isoline becomes smaller as the COVs of \( c \) and \( \phi \) decrease, that is, the
probability content of the joint PDF over the same failure set of the infinite slope decreases. Consequently, the probabilities of failure of the infinite slope will thus decrease although the limit state surfaces do not change. Similarly, Fig. 9 shows the joint PDF isolines of the shear strength parameters associated with different copulas for $\mu_{c,\phi} = -0.8$, $\text{COV}_c = 0.4$ and $\text{COV}_{\phi} = 0.2$. In comparison with Fig. 7, as the negative correlation between cohesion and friction angle becomes stronger, the PDF isoline becomes narrower, and the difference among the joint PDFs associated with different copulas becomes more significant. Like the results shown in Fig. 8, the probabilities of failure will become smaller although the limit state surfaces remain unchanged.

4. Summary and conclusions

This paper investigates the impact of copula selection on geotechnical reliability under incomplete probability information. Four copulas, namely Gaussian, Plackett, Frank, and No. 16 copulas, are selected to construct the joint PDF of cohesion and friction angle with given marginal distributions and correlation coefficient. The closed-form expressions of the probabilities of failure for the infinite slope and retaining wall using direct integration are derived. Two illustrative examples are presented to demonstrate the impact of copula selection on geotechnical reliability. Several conclusions can be drawn from this study:

1. The probabilities of failure for geotechnical structures associated with different copulas can differ considerably. The difference in probability of failure increases with decreasing probability of failure (or increasing nominal factors of safety). Significant difference in probabilities of failure associated with different copulas can be observed for relatively small COVs of shear strength parameters or a strong negative correlation between cohesion and friction angle. To the best of our knowledge, although one expects different results to be produced by different copulas, the magnitude and significance of these differences have not been reported in the literature.

2. The Gaussian copula, often adopted out of expedience without proper validation, may not capture the dependence structure between cohesion and friction angle properly. For the datasets considered, the Plackett copula and Frank copula are the best-fit copulas for modeling the dependence structure between cohesion and friction angle. For the four copulas considered, the Gaussian copula will result in a significant underestimation of the probability of failure for the infinite slope and retaining wall, which is unconservative for slope and retaining wall safety assessment. Thus, it is of practical importance to select the most appropriate copula to characterize the dependence structure of shear strength parameters when enough data are available.
Appendix A. Derivation of formulae for calculating probabilities of failure of the infinite slope and retaining wall using direct integration

By definition, the probabilities of failure for the infinite slope and retaining wall, \( p_f \), can be given by the following double integral,

\[
p_f = \int_{\mathcal{P}} f(c, \phi) \, dc \, d\phi \tag{A.1}
\]

where \( f(c, \phi) \) is the joint PDF of \( c \) and \( \phi \). Applying Eq. (3), Eq. (A.1) can be further expressed as

\[
p_f = \int_{\mathcal{P}} f_1(c) f_2(\phi) f_1(c, F_2(\phi); \theta) \, dc \, d\phi \tag{A.2}
\]

It is evident from Eq. (A.2) that the double integral may be time-consuming. For this reason, the first derivative of a copula function is employed, which is given by

\[
M(u_1, u_2; \theta) = \frac{\partial C(u_1, u_2; \theta)}{\partial u_2} \tag{A.3}
\]

By substituting Eq. (A.3) into Eq. (A.2), the double integral in Eq. (A.2) can be reduced to a single integral,

\[
p_f = \int_{\mathcal{P}} f_2(\phi) M(F_1(c), F_2(\phi); \theta) \, d\phi \tag{A.4}
\]

For further derivation, the expression of \( c \) in terms of \( \phi \) should be available, which can be obtained based on the limit state function \( g(c, \phi) = F_S(c, \phi) - 1 = 0 \). For the infinite slope example considered, \( c \) can be expressed as

\[
c = \gamma \tan \phi / \tan \pi \tag{A.5}
\]

Similarly, for the retaining wall example, by applying the Shengjin's formulae [10] to solve the cubic equation with respect to \( c \), \( c \) can be expressed as

\[
c = 0.5 \varphi_{\text{int}} \sqrt{K_a} H \left( \frac{1}{\gamma} \left[ \sqrt{\frac{6(W_1 \times \text{Arm}_1 + W_2 \times \text{Arm}_2)}{\varphi_{\text{int}} K_a}} \right] \right) \tag{A.6}
\]

Finally, substituting Eq. (A.5) into Eq. (A.4), one can obtain the probability of slope failure,

\[
p_f = \int_0^\pi f_2(\phi) M \left( F_1 \left( \gamma \tan \phi / \tan \pi \right), F_2(\phi; \theta) \right) \, d\phi \tag{A.7}
\]

Substituting Eq. (A.6) into Eq. (A.4), one can obtain the probability of retaining wall failure,

\[
p_f = \int_0^\pi f_2(\phi) M \left( 0.5 \varphi_{\text{int}} \sqrt{K_a} H \left( \frac{1}{\gamma} \left[ \sqrt{\frac{6(W_1 \times \text{Arm}_1 + W_2 \times \text{Arm}_2)}{\varphi_{\text{int}} K_a}} \right] \right), F_2(\phi; \theta) \right) \, d\phi \tag{A.8}
\]

in which \( \phi_0 \) is calculated by

\[
\phi_0 = \frac{\pi}{2} \left\{ \arctan \left( \sqrt{\frac{6(W_1 \times \text{Arm}_1 + W_2 \times \text{Arm}_2)}{\varphi_{\text{int}} K_a}} \right) \right\} \tag{A.9}
\]

Note that the probabilities of failure shown in Eqs. (A.7) and (A.8) are relatively simple if the first derivative of a copula is available. For convenience, Table 3 shows the first derivatives of the four copulas considered. When the copula parameters \( \theta \) are known, the probabilities of failure for the infinite slope and retaining wall can be efficiently evaluated using Eqs. (A.7) and (A.8), respectively.

References